மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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M.Sc. MATHEMATICS

II YEAR

DIFFERENTIAL GEOMETRY
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M.Sc. MATHEMATICS –II YEAR

SMAM42: DIFFERENTIAL GEOMETRY
SYLLABUS

UNIT-I:

Space curves: Definition of a space curve – Arc length – tangent – normal and binormal – curvature and torsion – contact between curves and surfaces- tangent surface- involutes and evolutes- Intrinsic equations – Fundamental Existence Theorem for space curves- Helices.

Chapter 1: Sections 1.1 - 1.9.

Unit II

Intrinsic properties of a surface: Definition of a surface – curves on a surface – Surface of revolution – Helicoids – Metric- Direction coefficients – families of curves- Isometric correspondence- Intrinsic properties.

Chapter 2: Sections 2.1 – 2.9

UNIT-III:

Geodesics: Geodesics – Canonical geodesic equations – Normal property of geodesics- Existence Theorems – Geodesic parallels – Geodesics curvature- Gauss- Bonnet Theorem – Gaussian curvature- surface of constant curvature.

Chapter 3: Sections 3.1 -3.9

UNIT-IV:

Non Intrinsic properties of a surface: The second fundamental form- Principal curvature – Lines of curvature – Developable - Developable associated with space curves and with curves on surface - Minimal surfaces – Ruled surfaces.

Chapter 4: Sections 4.1 – 1.8.

UNIT-V:

Differential Geometry of Surfaces: Compact surfaces whose points are umblics- Hilbert's lemma– Compact surface of constant curvature – Complete surface and their characterization – Hilbert's Theorem – Conjugate points on geodesics.

Chapter 5: Sections 5.1-5.7

Recommended Text

T.J. Willmore, An Introduction to Differential Geometry, Oxford University Press, (17th Impression) New Delhi 2002. (Indian Print).

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CONTENTS

UNIT-I:

Space curves: Definition of a space curve – Arc length – tangent – normal and binormal – curvature and torsion – contact between curves and surfaces- tangent surface- involutes and evolutes- Intrinsic equations – Fundamental Existence Theorem for space curves- Helices.

Chapter 1: Sections 1.1 -1.9.

1.1. Space curves:

A plane curve is usually specified either by means of single equation or else by a parametric representation

Example:

A circle with Centre at origin $(0,0)$ and radius a is specified in cartesian co-ordinate, (x, y) by single equation $x^2 + y^2 = a^2$ are else by the parametric representation

$x = a\cos u, y = a\sin u$

$0 \leq u \leq 2\pi$

Definition: Space curves

In three-dimensional Euclidian space E_3, A Single equation generally represent a surface and two equation are need to specify a curve.

∴ The curve appears as thus intersection of two surfaces represented by the two equations.

parametrically a curve may specify in Cartesian coordinates by equations

$x = x(u)$

$y = y(u)$

and $z = z(u)$.

where x, y, z are real valued functions of the real parameter 'u'. which is restricted to some interval.

Alternatively, in vector notation the curves are specified by vector value fiction

$\vec{r} = \vec{R}(\vec{u})$

Remark :1

A curve is defined by equation $F(x, y, z) = 0, G(x, y, z) = 0$. if F, G have its derivatives and if at least one of the Jacobian determinant,

$\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}$ is not zero at a point (x_0, y_0, z_0) on the curve.

It's known from the theory of implicit function that the equation $F = 0, G = 0$ Can be solved for two of the variables in terms of the 3^{rd} .

1.2. Definition of a space curve:

Function of class m.

Let I be a real interval and in a positive integer. A real value function f define on I is said to be of class m (or) to be c^m function. If f has an m^{th} derivative at every point of I and if this derivative is continuous on I.

Note:

c^m function as continuous m^{th} derivative when a function in infinitely differentiable we say it class ∞ or c^{∞}.

when a function is analytic we say it is of class w (or) c^w function.

Definition:

A vector valued function $R = (x, y, z)$ defined on I is said to be of class m if it has on m^{th} derivative at every point and if this derivative is continuous on I (or) equivalently if each of its components (x, y, z) is of class m.

Definition:

A function is frequently specified by the vector equation $R = (x, y, z)$ (or) equivalently by the 3 equation for the Cartesian components

$$x = x(u)$$
$$y = y(u)$$
$$z = z(u).$$

(if the derivative $\frac{dR}{du} = \dot{r} \neq 0$ Never vanishes on I or equivalently if $\dot{x}, \dot{y}, \dot{z}$ never vanishes simultaneously. Then the function is said to be regular. A regular vector valued function of class m is called a path of class m)

Definition:

Two paths R_1, R_2 of the same class m on I_1, I_2 are called equivalent if there exists a strictly increasing function ϕ of class m which maps I_1 onto I_2

(ie) $\phi: I_1 \overset{onto}{\rightarrow} I_2 \quad R_1 = R_2 \circ \phi$

The condition $R_1 = R_2 \circ \phi$ is equivalent to the three conditions

$$x_1(u) = x_2(\phi(u))$$
$$y_1(u) = y_2(\phi(u))$$
$$z_1(u) = z_2(\phi(u))$$

Note:

1. Any equivalence class of paths of class m determinds a curve of class m Class m

determines a curve of class m.

2. Any path R determines a unique curve and is called a parametric representati of the curve, the variable u being

called the parameter.

3. The equations,

$x = x$ (u)

$y = y(u)$

$z = z(u)$ are called parametric equation of the curve,

4. The mapping ϕ which relates two equivalent paths is called a change of parameter, It produces, the change in the manner of description of the curve the preserving sense

5.A curve of class m in E_3 as a Set of points in E_3 associtated with an equivalence class of regular parametric representation of class m involving one parameter.

6. When the function $R(u)$ is a linear, then equation $r = R(u)$ represent a straight line

7.Example of two equivalent representation Consider the circular helix is given by

$\bar{r} = (a\cos u, a\sin u, bu)$ where $0 \leq u \leq \pi$. …… (1)

Take, v $= \phi(u) = \tan\left(\frac{u}{2}\right)$

$\Rightarrow \tan^{-1}(v) = \frac{u}{2}$ $\Rightarrow 2\tan^{-1}(v) = u$ sub in (1)

$$\bar{r} = \left[a \cdot \frac{1 - \tan^2\left(\frac{u}{2}\right)}{1 + \tan^2\left(\frac{u}{2}\right)}, a\frac{2\tan\left(\frac{u}{2}\right)}{1 + \tan^2\left(\frac{u}{2}\right)}, bu\right]$$

$$\bar{r} = \left[a\frac{1-v^2}{1+v^2}, \frac{2av}{1+v^2}, b_2\tan^{-1}(v)\right] \qquad \text{………… (2)}$$

$$0 \leq u \leq \infty$$

we note that $\phi: I_1 \to I_2$

$[[0,\pi] \to\in [0,\infty]]$

$u = -\pi \Rightarrow u = \tan\pi/2 = \infty$

Where the function

$V = \phi(u) = \tan\left(\frac{u}{2}\right)$ is strictly increasing and onto

∴ The representation of equation (1) and (2) are equivalent.

Theorem 1:

Equivalence relation of a path is a proper equivalent relation on the path of the same class m.

Proof:

R_1, R_2, R_3 be any path of the same class m.

(i) Reflexive:

Define the identity function $i_\alpha: I_1 \to I_1$ is strictly increasing and onto.

Further, $R_1 = R_1 \circ i_\alpha$.

(i.e.) R_1 is equivalent to itself.

(ii) Symmetric:

Let R_1 be equivalent to R_2

T. P.T: R_2 is equivalent to R_1

From given R_1 is equivalent to R_2.

Then there exists a strictly increasing function

ϕ from I_1 onto I_2.

Such that $R_1 = R_2 \circ \phi$.

here ϕ : $1 - 1$, onto and increasing function

which $\Rightarrow \phi^{-1}$ exists and $\phi^{-1}: I_2 \to I_1$

which is strictly increasing and onto.

(ie) $R_2 = R_1 \circ \phi^{-1}$.

(ie) R_2 is equivalent to R_1.

(iii) Transitive:-

Given R_1 be equivalent to R_2 and R_2 be equivalent to R_3.

To prove that:- R_1 be equivalent to R_3.

From given R_1 is equivalent to R_2

there exists a strictly increasing function ϕ

from $I_1 \overset{\text{onto}}{\to} I_2$

Such that $R_1 = R_2 \circ \phi$. ……….. (1)

Also given R_2 is equivalent to R_3.

$\Rightarrow$ There exists a strictly increasing function. $\psi: I_2 \overset{\text{onto}}{\to} I_3$ ……….. (2)

Such that, $R_2 = R_3 \circ \Psi$

ϕ and ψ are strictly increasing function and onto. From I_1 onto I_2 and I_2 to I_3 respectively

$\Rightarrow \psi \circ \phi$ is also a strictly increasing Function and onto from I_1 to I_3.

$\therefore$ using (2) in (1).

$(1) \Rightarrow R_1 = (R_3 \circ \psi) \circ \phi$

$\Rightarrow R_1 = R_3 \circ (\psi \circ \phi)$

with $\psi \circ \phi$ is strictly increasing function from I_1 onto I_3.

(i.e.) The path R_1 is equivalent to R_3.

∴ equivalence relation of path is a proper equivalent relation on the path of the same class m.

Note:

(i) Not every property of a path is a property of the curve.

(ii) The property or the curve are those Which are common to all parametric representation.

(iii) It the function $R(u)$ is a linear then the equation $r = R(u)$ represents a straight line.

1.3. Arc Length:

1.Distance between two points in Euclidean Space

The Distance between two points $\bar{r}_1 = (x_1, y_1, z_1)$ & $\bar{r}_2 = (x_2, y_2, z_2)$ in Euclidean space is the number.

(i.e) $$\begin{aligned} |\bar{r}_1 - \bar{r}_2| &= \sqrt{(\bar{r}_1 - \bar{r}_2)^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \end{aligned}$$

This distance in space will be used to define distance along type a curve of class $m \geq 1$.

2. **Arc**

It given a path $\bar{r} = \bar{R}(u)$, and two numbers $a, b(a < b)$ in the range of the parameter then the path $\bar{r} = \bar{R}(u)(a \leq u \leqslant b)$ is an arc of the original path Joining the points corresponding to a & b.

3. **Length of polygon**:

Any subdivision Δ of the interval (a, b) by points $a = u_0 < u_1 < u_2 < \cdots < u_n = b$

The correspondence the length

$$L_\Delta = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})|$$

of the polygon inserted to the arc by joining successive points on it.

Addition of further points, subdivision increases the length of polygon. Because two sides of the triangle of are together greater than 3rd .

4) The length of are to be trouper bonded of L_Δ taken over all possible sub divisions of (a, b). This upper bound is always Finite.

∴ $L_\Delta \leqslant \sum_{i=1}^{n} \int_{u_{i-1}}^{u_{i\to 1}} |\dot{R}(u)| du$.

Definition: Arc length

If $a < c < b$ then the arc length from a to b is some of the arc length From a to c and from c to b

$S = S(u)$

The arc length from a to any point u The are length from u_0 to $u = S(u) - S(u_0)$

S is a function of the same class as the curve

$\therefore s = S(u) = \int_a^u |\dot{R}(u)| du$ in terms of a cartesian parametric represent

$$S = S(u) = \int_a^{a} \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} du$$

Note:

(i) The equation $\dot{s} = |(\bar{r})|$ in cartesian parametric representation is

$\dot{s} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ (or)

$ds^2 = dx^2 + dy^2 + dz^2$.

(ii) The function S is the change of parametric from s to u

$\therefore u = \phi(s)$

(iii) The curve parametrized with respect to S is $r^{\rightarrow} = R^{\rightarrow}(\phi(s))$

Example 1:

Obtain the equations of the circular helix, $\bar{r} = (a\cos u, a\sin u, bu), -\infty < u \leqslant \infty$ where a>0 refer to S as parameter and show that the length of one complete from turn of the helix x is $2\pi C$. Where $c = \sqrt{a^2 + b^2}$

Solution:

Given, $\bar{r} = (a\cos u, a\sin u, bu)$ ……………(1)

(i) To find the equation of the circular helix with

$\therefore x = a\cos u \Rightarrow \dot{x} = dx = -a\sin u$ parameter s.

$y = a\sin u \Rightarrow \dot{y} = dy = a\cos u$

$z = bu \Rightarrow \dot{z} = dz = b$

Arc length $= s = S(u) = \int_n^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} du$ (given a>0)

$$= \int_0^4 \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2} du$$

$$= \int_0^u \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2} du$$

$$= \int_0^4 \sqrt{a^2 + b^2} du$$

given $c = \sqrt{a^2 + b^2}$

$$s = \int_0^u c du$$

$$= c \int_0^u du$$

$$= c[u]_0^u$$

$$s = cu$$

$$\Rightarrow \frac{s}{c} = u \quad \text{sub in (1)}$$

$(1) \Rightarrow \bar{r} = \left(a\cos\frac{\beta}{c}, a\sin\frac{\beta}{c}, b\frac{b}{c}\right).$

To show that the length of one Complete turn of the helix $= 2\pi c$

where $c = \sqrt{a^2 + b^2}$

The Range or corresponding to one Complete turn of the helix is

$u_0 \leq u \leq u_0 + 2\pi$

∴ The Length of one furn to the circular helix equal to $u_0 + 2\pi$

length of the $arc = \int_{u_0}^{u_0+2\pi} \sqrt{a^2 + b^2} du$

$= \sqrt{a^2 + b^2}[u]_{u_0}^{u_0+2}$

$$= \sqrt{a^2 + b^2}[u]_{u_0}^{u_0+27}$$

$$= c[u_0 + 2\pi - u_0]$$

$= 2\pi c$

Example :2

Find the length of the curve given as the intersection of the surfaces.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $x = a\cosh\left[\frac{z}{a}\right]$ from the point $(a, 0,0)$ to the point (x, y, z)

Proof:

given surfaces.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots\ldots (1)$$

$x = \text{a} \cosh[\frac{z}{a}]$ ………. (2)

We know that,

The length of the curve $= \int_a^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$ ………….(3)

put $\frac{z}{a} = t$ in (2)

$z = at$

(2) $\Rightarrow x = a\cosh t$ sub in (1).

(2) $\Rightarrow \frac{a^2\cosh^2 t}{a^2} - \frac{y^2}{b^2} = 1$

$\cosh^2 t - \frac{y^2}{b^2} = 1$

$\Rightarrow \cosh^2 t - 1 = y^2/b^2$

$\Rightarrow \sinh^2 t = y^2/b^2$

$\Rightarrow b^2\sinh^2 t = y^2$

$\Rightarrow y = b\sinh t \quad => \sinh t = y/b$

$$\left.\begin{array}{ll} x = a\cosh t & \dot{x} = a\sinh t \\ y = b\sinh t & \dot{y} = b\cosh ht \\ z = at & \dot{z} = a \end{array}\right\}$$

Sub in eqn (2).

The length of the curve $= \int_0^t \sqrt{a^2\sin h^2 t + B^2\cos t + a^2}\Big]\, dt$

$= \int_0^t \sqrt{a^2(\sinh^2 t + 1) + b^2\cos h^2 t dt}$

$= \int_0^t \sqrt{a^2\cos h^2 t + b^2\cos h^2 t} dt$

$=\cdot \int_0^t \sqrt{a^2 + b^2}\cos ht \cdot dt$

$= \sqrt{a^2 + b^2}[\sinh t]_0^t$

$= \sqrt{a^2 + b^2}\sin ht.$

$= \sqrt{a^2 + b^2}\ (y/b)$

1.4.Tangent normal and bi-normal:

$\bar{r}$ = The position vector of a point on a curve and also as the function Symbol of a path which represents the curve a curve represented by the equation.

$\bar{r} = \bar{r}(u)$

$\bar{R}$ = The position vector of a current point in space not necessarily lying on the curve.

Let r be a curve of class ≥ 1 and let (P, Q) be two neighbouring pts of the curve.

Definition: Unit tangent vector

Let r be represented by the equation

$\bar{r} = \bar{r}(u)$

and let P and Q have parameters u_0 and u Since r has class ≥ 1.

$\therefore \bar{r}(u) = \bar{r}(u_0) + (u - u_0)\dot{r}(u_0) + 0(u - u_0)$ ……………(1)

where $o(u - u_0) = \frac{(u-u_0)^2}{2!}\frac{\dddot{}}{r}(u_0) + \frac{(u-u_0)^3}{3!}\frac{\dddot{}}{r}(u_0) + \cdots$ ....

$$\therefore \lim_{u\to u_0} \frac{\bar{r}(u)-\bar{r}(u_0)}{|\bar{r}(u)-\bar{r}(u_0)|} = \frac{\dot{\bar{r}}(u_0)}{|\dot{\bar{r}}(u_0)|}$$

$$\left[\because \lim_{u\to u_0} \frac{\bar{r}(u)-\bar{r}(u_0)}{u-u_0} = \dot{\bar{r}}(u_0)\right]$$

(i.e) The unit vector along the chord PQ tends to a unit vector at P as $Q \to P$

This is called the unit tangent vector to r at P. and it is denoted by $\bar{t}$.

$$\therefore \bar{I} = \frac{\dot{\bar{r}}(u_0)}{|\dot{\bar{r}}(u_0)|}$$

$$= \frac{\dot{r}}{\dot{s}} \left[\because \dot{r}(u_0) = \frac{dr}{du} \ \& \ |\dot{\bar{r}}(u_0)| = \frac{ds}{du}\right]$$

$$= \frac{dr}{ds}$$

Note:

(i) $\bar{t}$ like the curve is oriented in that its points in the direction of increasing $\bar{s}$

(ii) The line through P parallel to $\bar{t}$ is called the tangent line to γ at P.

(iii) It $\bar{R}$ is any point on this line, the vector from the the vector from the pt of contact,

P to R is called a tangent vector to ν at P.

(iv) tangent line is a unique line which approximates to the curve to the 1^{st} order near P more preciously there is a unique linear function $L(w)$.

Such that,

$L(u) = \bar{r}(u) + 0(u-u_0)$ as $u \to u_0$

$= r(u_0) + (u-u_0)\dot{\bar{r}}(u_0)$

(v) the unit tangent vector $\bar{t} = \bar{r}'$.

Definition: Osculating plane

Let γ be a curve of class ≥ 2. and let (P, Q) be two neighbouring p ts on ν then the limiting position as $Q \to P$. If that plane which contains the tangent line at p and the point Q is called the osculating plane of γ at P.

Theorem 1:

Show that when a curve is analytic we obtain a definite oscillating plane at a point of inflection P unless the curve is a straight line.

Proof:

case (i)

Let P is not a point of inflection

(ie) $\bar{r}'' \neq 0$ at $'P'$ ……….(1)

Let the curve gamma be a parameter

$w.r$ to the are length s.

Let P, Q be two neighboring points on γ w.r. to the parameters 0 and S.

Consider the plane containing the tangent

vector to γ at P and the point Q.

We know that, $\overline{PQ} = \overline{OQ} - \overline{OP}$

$= \bar{r}(s) - \bar{r}(0)$

$\overline{PQ} = \bar{R} - \bar{r}(0)$ .... (2)

$\overline{pm}, \bar{r}'(0)$ & $\overline{PQ}$ are coplanar.

(i.e.) The equation of the plane is

$[P\bar{Q}, \bar{r}'(0), P\bar{m}]$

(i.e.) $[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}(s) - \bar{r}(0)] = 0,$ ....(3)

We know that, by Taylor's theorem,

$\bar{r}(s) = \bar{r}(0) + \frac{s\bar{r}'(0)}{1!} + \frac{s^2}{2!}\bar{r}''(0) + 0(s)$ as $s \to 0$.

$\therefore \bar{r}(s) - \bar{r}(0) = \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + 0(s)$ as $s \to 0$

$\therefore (2) \Rightarrow$

$\left[\bar{R} - \bar{r}(0), \bar{r}'(0), \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + o(s)\right] = 0$ (ie) $\left[\bar{R} - \bar{r}(0), \bar{r}'(0), \frac{s}{1!}\bar{r}'(0)\right] +$

$\left[\bar{R} - \bar{r}(0), \bar{r}'(0\frac{s^2}{2!}r''(0)\right]$

$+[\bar{R} - \bar{r}(0), \bar{r}'(0), 0(s)] = 0$ as $s \longrightarrow 0$

(ie) $\left[0 + \frac{s^2}{2!}[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] + 0 = 0\right.$

(ie) $[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0$

is the equation of the required osculating plane provided that,

$\bar{r}'(0), \bar{r}''(0)$ are $L.I.$

Since $|\bar{t}| = 1$

Suppose $\bar{t} = \bar{r}'$,

$\Rightarrow |\bar{r}'| = 1$

$\Rightarrow \bar{r}' \cdot \bar{r}' = 1$

Diff w.r. to 's'.

$\Rightarrow \bar{r}' - \bar{r}^n + \bar{r}'' - \bar{r}' = 0$

$\Rightarrow 2\bar{r}' \cdot \bar{r}'' = 0.$

$\Rightarrow \bar{r}' \cdot \bar{r}'' = 0$

(i.e) $\bar{r}'$ & $\bar{r}''$ are perpendicular to each other

(i.e.) r' & $\bar{r}''$ are L.I.

Hence the eqn of the osculating plane

$[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0$ …………(4)

Case(ii)

Let P is a point of inflection

(i.e.) $r''(0) = 0$ at p … … . (5)

Let γ is not a straight line and γ is analytic.

Since $|\bar{t}| = 1 \Rightarrow |\bar{r}'| = 1$

(ie) $|\bar{r}'| \cdot |\bar{r}'| = 1$

(ie) $|\bar{r}'|^2 = 1$

$\Rightarrow \bar{r}' \cdot \bar{r}' = 1$

Diff w,r.to ' s '

$\Rightarrow \bar{r}' \cdot \bar{r}'' + \bar{r}'' \cdot \bar{r}' = 0$

$\Rightarrow 2\bar{r}' \cdot \bar{r}'' = 0$

$\Rightarrow \bar{r}' \cdot \bar{r}'' = 0$ at ' p '

Again Diff w.r.to 's'

$\Rightarrow \bar{r}' \cdot \bar{r}''' + \bar{r}'' \cdot \bar{r}'' = 0 \ldots \ldots \ldots \ldots (6).$

$\Rightarrow r' \cdot \bar{r}''' = 0$ at P

(i.e) $r'\ perpendicular\ r'''$

(i.e) $\bar{r}'$ & $\bar{r}'''$ are L.I.

Again Diff w.r.to 's'

$\therefore \bar{r}' \cdot \bar{r}''' + \bar{r}'' \cdot \bar{r}''' = 0$

$\Rightarrow \bar{\tau}' \cdot \bar{r}''' = 0$ at $'p'$

similarly continuing this process, we get,

$\bar{r}'\bar{r}^k = 0$ at $p \ldots \ldots \ldots . (7)$

where r^k represent the non-zero derivative. of $\bar{r}$ at P for $k \geq 2$.

Further,

given the curve γ is analytic

$\therefore r''(s) = \bar{r}''(0) + \frac{s}{1!}\bar{r}'''(0) + \frac{s^2}{2!}\bar{r}''''(0) + ds$ as $s \to 0$

$\therefore \bar{r}'''(s) = 0, \forall s.$

$\Rightarrow \bar{r}''(s) = \text{constant}$

$\Rightarrow \gamma$ is a $straight$ line

which is a contradiction.

$\therefore$ from(6),

$\bar{r}^{(k)}(0) \neq 0$, for $k = 2,3 \dots (k-1)$

$$\bar{r}(s) = \bar{r}(0) + \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(b) + \cdots + \frac{g^k}{k!}\bar{r}^k(0)$$

$$\therefore \bar{r}(s) - \bar{r}(0) = \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \cdots + \frac{s^k}{k!}\bar{r}^k(\theta)$$

$$= \frac{s}{1!}\tau'(0) + \frac{s^k}{k!}r^k(0) + o(s) \text{ as } s \to 0$$

We know that, the equation of the plane is,

$$[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}'(s) - \bar{r}'(0)] = 0$$

(i.e) $\left[\bar{R} - \bar{r}(0), \bar{r}'(0), \frac{s}{1!}\bar{r}'(0) + \frac{s^k}{k!}\bar{r}^k(0) + 0(s)\right]$

(i.e) $\left[\bar{R} - \bar{r}(0), \bar{r}'(0), \frac{s}{1!}\bar{r}'(0)\right]$

$$+\left[\bar{R} - \bar{r}(0), \bar{r}'(0), \frac{s^k}{k!}\bar{r}^k(0)\right]$$

$$+[\bar{R} - \bar{r}(0), \bar{r}'(0), o(s)] = 0$$

The eq of the osculating plane become

$$[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}^k(0)] = 0$$

Example 1:

Consider the curve γ is defined by $\bar{r}(u) = \left(u, e^{-\frac{1}{u^2}}, 0\right), u < 0, \bar{r}(u) = \left(u, 0, e^{-\frac{1}{u^2}}\right), u > o, \bar{r}(0) = (0,0,0)$ · Show that at a point of inflection even a curve of class infinity need not possess an osculating plane.

Proof:

We know that osculating plane at all points with parameter $u < 0$.

(i.e.) $\bar{r}(u) = \left(u, e^{-1/u^2}, 0\right)$

$$\Rightarrow \dot{r}(u) = \left(1, \frac{2}{u^3}e^{-1/u^2}, 0\right)$$

$$\Rightarrow \ddot{\bar{r}}(u) = \left(0, \left[-\frac{6}{u^4}e^{-1/u^2} - \frac{4}{u^6}e^{-1u^2}\right], 0\right)$$

$\therefore$ The eq of the osculating plane is if $u < 0$.

$$\begin{vmatrix} x-u & y-e^{-1/u^2} & z-0 \\ 1 & \frac{2}{u^3}e^{-1/u^2} & 0 \\ 0 & \frac{-6}{u^4}e^{-1/u^2} - \frac{4}{u^6}e^{-1/u^2} & 0 \end{vmatrix} = 0$$

$$\Rightarrow z\left[-\frac{2}{u^4}e^{-1/u^2}\left[3+\frac{2}{u^2}\right]\right] = 0$$

$\Rightarrow z = 0$

if $u < 0$ then $z = 0$ [The equation of the osculating plane]

Similarly, the equation of the osculating plane at all points on γ with $u > 0$ is $y = 0$

To find the limit $\dot{\bar{r}}(0)$ for $u < 0$.

$$\therefore \dot{r}(0) = \lim_{u\to 0^-} \frac{\bar{r}(u) - \bar{r}(0)}{u-0}$$

$$= \lim_{u\to 0^-} \frac{\left(u, e^{-1/u^2}, 0\right) - (\bar{0}, 0,0)}{u}$$

$$= \lim_{u\to 0^-} \frac{\left(u, e^{-1/u^2}, 0\right)}{u}$$

$$= \lim_{u\to 0^-} \left(1, \frac{e^{-1/u^2}}{u}, 0\right) \text{ if } u < 0 \ \dot{\bar{r}}(0) = (1,0,0)$$

Similarly $\dot{r}(0) = (1,0,0)$ if $u > 0$.

Now, To find the limit $\ddot{r}(0)$ for $u < 0$

$$\ddot{\bar{r}}(0) = \lim_{u\to 0^-} \frac{\dot{\bar{r}}(u) - \dot{\bar{r}}(0)}{u-0}$$

$$= \lim_{u\to 0^-} \frac{\left[1, \frac{2}{u^3}e^{-1/u^2}, 0\right] - [1,0,0]}{u}$$

$$= \lim_{u\to 0^-} \frac{\left[0, \frac{2}{u^3}e^{-1/u^2}, 0\right]}{u}$$

$$= \lim_{u\to 0^-} \left[0, \frac{2}{u^4}e^{-1/u^2}, 0\right]$$

$\ddot{r}(0) = (0,0,0)$ if $u < 0$

Similarly $\dot{r}(0) = (0,0,0)$ if $u > 0$.

$\therefore$ we get $\ddot{r}(0)$ exists and $\ddot{\vec{r}}(0) = (0,0,0)$

Hence $u = 0$ is a point of inflection for $k \geq 2$

Extending like we get

$\bar{r}^k(0) = \bar{0}$.

Thus at a point of inflection, even a curve of class infinity need not possess an osculating plane.

Example 2:

Show that if a curve is given in terms of a general parameter ' u ' then the eqn of the osculating plane corresponding to

$[R - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0$ is $[R - \bar{r}, \dot{r}, \ddot{r}] = 0$

Solution:

$$\bar{t} = \bar{r}' = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/du}{ds/du}$$

$$= \frac{\dot{\bar{r}}}{\dot{s}}$$

$$\therefore \bar{r}' = \frac{\dot{\bar{r}}}{\dot{s}} \; \frac{u}{v} = \frac{vdu - udv}{v^2}$$

$\therefore$ Diff w.r.t. ' s '

$$\therefore \bar{r}'' = \frac{\dot{s}\ddot{r} - \dot{\bar{r}}\ddot{s}}{\dot{s}^2}\frac{du}{ds} \left(\because \frac{du}{ds} = 1\right)$$

$$\text{given}, \Rightarrow \bar{r}'' = \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{(\dot{s})^2}$$

$$\left[\bar{\bar{R}} - \bar{r}(0), \bar{r}''(0), \bar{r}''(0)\right] = 0$$

$$\Rightarrow \left[R - \bar{r}(0), \frac{\dot{\bar{r}}}{\dot{s}}, \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^2}\right] = 0$$

$$\Rightarrow \left[R - \bar{r}(0), \frac{\dot{\bar{r}}}{\dot{s}}, \frac{\dot{s}\ddot{\bar{r}}}{\dot{s}^x}\right] - \left[R - \bar{r}(0), \frac{\dot{r}}{\dot{s}}, \frac{\dot{r}\ddot{s}}{\dot{s}^2}\right] = 0$$

$$\Rightarrow \left[R - \bar{r}(0), \frac{\dot{\bar{r}}}{\dot{s}}, \frac{\ddot{\bar{r}}}{\dot{s}}\right] - \left[R - \bar{r}(0), \frac{\dot{r}}{\dot{s}}, \frac{\dot{\bar{r}}\ddot{s}}{\dot{s}^2}\right] = 0$$

$$\Rightarrow \frac{1}{\dot{s}^2}[R - \bar{r}(0), \dot{r}, \ddot{r}] - 0 = 0$$

$\Rightarrow [R - \bar{r}(\infty), \dot{r}, \ddot{r}] = 0$ in the equation of the osculating plane.

Example 3:

Find the equation of the osculating plane at a general paint on the cubic curve given by $\bar{r} = (u, u^2, u^3)$.

Show that the osculating plane at any three points of the curve meet at a point lying in the plane determined by this three points.

Proof:

We know that, the equation of the osculating plane at any point.

$[R-\bar{r}(0),\dot{r},\ddot{r}]=0$ ………. (1)

$g_n.r=(u,u^2,u^3)$

Therefore, $\dot{r}=(1,2u,3u^2)$ and $\ddot{r}=(0,2,6u)$and $R=(X,Y,Z)$ sub in eqn(1)

∴ (1) ⇒ $[(x-u),(y-u^2),(z-u^3)],(1,2u,3u^2\ ),(0,2,6u)]=0.$

$$\begin{vmatrix} x-u & y-u^2 & z-u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}=0$$

$$\Rightarrow (x-u)(12u^2-6u^2)-(y-u^2)[6u-0]+(z-u^3)$$
$$(2-0)=0$$
$$\Rightarrow (x-u)(6u^2)-(y-u^2)(bu)+(z-u^3)(2)=0$$
$$\Rightarrow 6u^2x-6u^3-6uy+6y^3+2z-2u^3=0.$$
$$\Rightarrow 6u^2x-6uy+2z-2u^3=0$$

$$\Rightarrow 3u^2x-3uy+z-u^3=0.$$

equation of the osculating plane to the cubic curve.

If u_1,u_2,u_3 are three distinct values of the parameter.

∴ The osculating plane at this points are linearly Independent and the plane meat at a point (x_0,y_0,z_0).

∴ The parameters u_1,u_2,u_3 satisfying the condition, $u^3-3u^2x_0+3uy_0-z_0=0$-(2)

If $lx+my+nz+p=0$, is a equation of the plane passing through the 3 points

Then the parameters satisfy the conditions

$lu+mu^2+nu^3+p=0$ ………. (3)

Since the equation (3) have 3 distinct roots and we have $n\neq 0$

∴ The corresponding coefficient are

equation (2) and (3) .

we get.

$$\frac{1}{n}=\frac{-3x_0}{m}=\frac{3y_0}{l}=\frac{-z_0}{p}$$

$$\left.\begin{aligned}&\Rightarrow l=3ny_0\\&\Rightarrow m=-3nx_0\\&\Rightarrow p=-nz_0\end{aligned}\right\}\text{ sub in (3)}$$

∴ (3) ⇒

The eqn of the plane is

$3ny_0x - 3nx_0y + nz - nz_0 = 0$

$\div n \Rightarrow 3y_0x - 3x_0y + z - z_0 = 0.$

Definition: Normal plane

The normal plane at a print p on a curve is that plane through p which is orthogonal to the tangent at p

Definition: Principle Normal

The principle Normal at p is a line of intersection of the Normal plane and the osculating plane at P. A unit vector along the principle Normal is denoted by $\bar{n}$.

Definition: curvature

The arc rate at which the tangent changes direction as P moves along the curve is the curvature of the curve, and is denoted by K (kappa)

(ie) $|\kappa| = |\bar{t}'| = \left|\frac{dt}{ds}\right|$

(i.e.) Arc rate at which $\bar{t}$ Exchanges direction

Theorem 2:

Prove that $\bar{t}' = k\bar{n}$

Proof:

We know that $\bar{r}' = \bar{E}$ &$|E| = 1$

$\bar{r}' \cdot \bar{r}'' = 1$

Diff w. r. to 's'.

$$\bar{r}' \cdot \bar{r}'' + \bar{r}'' \cdot \bar{r}'' = 0$$
$$\Rightarrow 2\bar{r}' \cdot \bar{r}'' = 0$$
$$\Rightarrow \bar{r}' \cdot \bar{r}'' = 0.$$
$$\Rightarrow \bar{r}' \cdot \bar{r}'' = 0.$$
$$\Rightarrow \bar{t} \cdot \bar{r}'' = 0.$$
$$\Rightarrow \bar{t} \text{ perpendicular } \bar{r}''$$

$\therefore r''$ lies in the Osculating plane.

$\therefore \bar{r}''$ is proprotional to $\bar{n}$

$\therefore \bar{r}'' = \pm k\bar{n}$

$\Rightarrow | \bar{r}''|| = +k\bar{n}.$

$\Rightarrow \bar{t}' = +k\bar{n},\ \bar{t}$ is called the curvature of the vector.

Theorem 3:

A Necessary and sufficient condition that A ere curve be a straight line is that $\kappa = 0$ at all points (or) Show that a curve is a straight line iff the curvature $k = 0$

Proof:

Necessary part:

Assume that the curve be a straight line.

To prove that : The curvature $\kappa = 0$.

We know that, the vector equation of the straight line is

$\bar{r} = \bar{a} + s\bar{b}$

where $\bar{a}$ & $\bar{b}$ are constants

and S is a parameter

Diff w.r. to ' s '.

$\Rightarrow \bar{r}' = 0 + \bar{b}$
$\Rightarrow \bar{r}' = \vec{b}$
$\Rightarrow \bar{t} = \bar{b}$

Again diff w.r.to 's'

$\Rightarrow t' = 0$

$\Rightarrow k\bar{n} = 0\ [\bar{t}' = k\bar{n}]$

$k = 0 (\because n \neq 0)$.

Sufficient part :

Assume that $\kappa = 0$ at all points

To prove that: a curve in a straight line.

As $k = 0$
$\Rightarrow \kappa\bar{n} = 0$
$\Rightarrow \bar{t}' = 0$
$\Rightarrow \bar{r}'' = 0$.

Integrate w.r.to 's'.

$\Rightarrow \bar{r}' = \bar{b}$ where $\bar{b} =$ constant.

Again Integrate w. r. to ' s '

$\Rightarrow \bar{r} = s\bar{b} + \bar{a}$ where $\bar{a} =$ constant

This is represent the eqn of a straight line.

Definition: Binormal line

The Binormal line at D is the normal in a direction orthogonal to the osculating plane

The sense of the unit vector $\bar{b}$ along the Binormal is chosen show that $\bar{t}, \bar{n}, \bar{b}$ form right hand system of axis.

(i.e.) $\bar{b} = \bar{t} \times \bar{n}$

Definition: Torsion

As P moves along a curve the are rate at which the osculating pane turns above the tangent is called the torsion of the curve and is denote by τ. $\tau = |\bar{b}'|$

(or) $\vec{b}' = -\tau\bar{n}$.

Theorem 4:

Prove that $b' = -\tau\bar{n}$.

Proof:

We know that, $|\bar{b}| = 1$

$\Rightarrow \bar{b} \cdot \bar{b} = 1$

"Diff w.r.to 's'.

$\Rightarrow \bar{b}' \cdot \bar{b} + \bar{b} \cdot \bar{b}' = 0$

$\Leftrightarrow 2\bar{b}' \cdot \bar{b} = 0$

$\Rightarrow \bar{b}' \cdot \bar{b} = 0$

$\Rightarrow \bar{b}' \perp^{r} \bar{b}$

$\Rightarrow \bar{b}' \& \bar{b}$ are L.I

$\Rightarrow \bar{b}'$ lines in the osculating plane

Also $\bar{b} \perp^{r} \bar{t} \Rightarrow \bar{b} - \bar{t} = 0$

Differentiate with respect to 's'.

$\Rightarrow \bar{b}' \cdot \bar{t} + \bar{b} \cdot \bar{t}' = 0.$

$\Rightarrow \bar{b}' \cdot \bar{t} + \bar{b} \cdot \bar{r}'' = 0$

$\Rightarrow \bar{b}' \cdot \bar{t} = 0$

$\Rightarrow b' \perp^{r} \bar{t}$

$\Rightarrow \bar{b}'$ & t are L.I.

$\Rightarrow \bar{b}'$ lies in the osculating plane & is proportional to n^{-}

$\Rightarrow \bar{h}'$ is $||^{le}$ to $\bar{n}$

(i.e.) $\bar{b}' = \pm\tau\bar{n}$

$\Rightarrow \bar{b}' = -\tau\bar{n}$

Note:

(1) Torsion is regarded as positive when the rotation of the osculating plane has S increasing is in the direction of a Right find travelling in the direction of $\bar{t}$.

(2) A Torsion τ is determined both in magnitude and sign.

(3) The curvature κ is determined only in magnititude.

Theorem 5:

Let γ be a curve for which $\bar{b}$ various differentially with are length then a necessary and sufficient condition that a curve γ be a plane curve is that $\tau = 0$ at all points.

Proof:

Given γ be a curve for which $\bar{b}$ various differentially with arc length.

Necessary part:

Assume that $\gamma = 0$ at all points

To prove that: $\tau = 0$ at all points.

from our assumption, the osculating plane at any point is Just the plane containing the curve

(i.e.) The osculating plane is fixed.

$\bar{b}$ is a constant vector.

$\Rightarrow \bar{b}' = 0.$

$\Rightarrow -\tau\, \bar{n} = 0\ (\bar{n} \neq 0)$

$\Rightarrow -\tau = 0$

$\Rightarrow \tau = 0$ at all points.

Sufficient part:

Assume that $\tau = 0$ at all points.

To prove that: The curve α is plane curve.

From our assumption, $\tau = 0$.

$\Rightarrow -\tau\bar{n} = 0$

$\Rightarrow \bar{b} = 0$

Integrate with respect to s

$\bar{b} =$ constant.

We know that, $\bar{t} \cdot \bar{b} = 0$

(ie) $\bar{r}'\bar{b} = 0.$

$\Rightarrow \bar{r}'\bar{b} + \bar{r}\bar{b}' = 0 \quad [\because \bar{b}' = 0]$

$\Rightarrow (\bar{r} \cdot \bar{b})' = 0.$

$\Rightarrow \bar{r} \cdot \bar{b} =$ constant.

(i.e.) $\bar{r}, \bar{b} = c$ .... (1)

where $\bar{b} = (b_1, b_2, b_3)$ &

$\bar{r} = (x(s), y(s), z(s))$

$\therefore$ (1) $b_1 x(s) + b_2 y(s) + b_3 z(s) = c =$ constant.

$\Rightarrow$ This condition shows that the pt

$(x(s), y(s), z(s))$ lies in the plane.

$\therefore b_1x + b_2y + b_3z = c$

$\therefore \gamma$ is lies in the plane curve.

Example 4:

Prove that $[\bar{r}', \bar{r}'', \bar{r}'''] = \kappa^2\tau$

Solution:

We know that, $\bar{t} = \bar{r}' \& \ \bar{t}' = \bar{r}''$.

$\therefore \bar{r}' \times \bar{r}'' = \bar{t} \times \bar{t}'$

$= t \bar{\times} \kappa\bar{n}$

$= \kappa(\bar{t} \times \bar{n})$

$\bar{r}' \times \bar{r}'' = \kappa\bar{b}$

Differentiate with respect to s

$\bar{r}'' \times \bar{r}'' + \bar{r}' \times \bar{r}'' = k'\bar{b} + k'\bar{b}'$

$'' \times \bar{r}''' - k'\bar{b} - k'\tau\bar{n}.$

$0 + \bar{r}' \times \bar{r}''' = k'\bar{b} - k'\tau\bar{n}.$

$\Rightarrow \bar{r}' \times \bar{r}''' = k'\bar{b} - k\tau\bar{n}$

Multiply both sides Secularly by $\bar{r}''$ '

$\Rightarrow \bar{r}''. [\bar{r}' \times \bar{r}'''] = \bar{r}''. [k'\bar{b} - k\tau\bar{n}]$

$\Rightarrow -[\bar{r}', \bar{r}'', \bar{r}'''] = E' \cdot [k'\bar{b} - k\tau\bar{n}]$

$= k\bar{n}[k'b - k\tau\bar{n}].$

$-[\bar{r}', \bar{r}'', \bar{r}'''] = 0 - k^2\tau$

(i.e.) $[\bar{r}', \bar{r}'', \bar{r}''] = k^2\tau.$

Example 5:

Show that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$ is a necessary and sufficient condition that the curve be a plane curve.

Proof:

Necessary part:

Assumes that the curve be a plane curve.

To prove that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$.

We know that, $\bar{r}' = \frac{d\bar{r}}{ds}$

$= \frac{d\bar{r}}{du} \cdot \frac{du}{ds}$

$= \dot{\bar{r}}. du/ds$ ………..(1)

Diff equation 1 with respect to s,

$\therefore \bar{r}'' = \dot{\bar{r}}u'' + \frac{d\dot{r}}{ds}u'$

$= \dot{\bar{r}}u'' + \frac{d\dot{r}}{du} \cdot \frac{du}{ds}u'$

$= \dot{r}u'' + \ddot{r}u'u'$

$\vec{r}' = \dot{r}u'' + \dot{\vec{r}}(u')^2$ …………..(2)

Diff (2) w.r.to ' s '.

$r''' = \dot{\bar{r}}u''' + \frac{d\dot{\bar{r}}}{ds}u'' + \ddot{\vec{r}}q(w')(u'') + \frac{d\dot{\vec{r}}}{ds}(u')^2$

$\tilde{r}''' = \hat{r}u''' + \frac{d\dot{r}}{du} \cdot \frac{du}{ds}u'' + 2\ddot{\vec{r}}u'u'' + \frac{d\ddot{r}}{du}\frac{du}{ds}(u\ \bar{r}''' = \dot{r}u''' + \ddot{r}u'u'' + \alpha\ddot{r}u'u'' + \ddot{r}(u')^3.$

$\bar{r}''' = \dot{r}u''' + 3\ddot{r}u'u'' + \dddot{r}(u')^3$

consider,

$[\bar{r}', \vec{r}'', \vec{r}'''] = \{\dot{r}u', [\dot{r}u'' + \ddot{\vec{r}}(u')^2],$

$[\dot{\bar{r}}u''' + 3\ddot{r}u'u'' + \dddot{r}(u')^3]$

$= [\dot{\bar{r}}u' \cdot \dot{\bar{r}}u'', [\dot{\bar{r}}u''' + 3\dot{\vec{r}}u'u''' + \ddot{r}(u')^3]$

$+[\dot{r}u', \ddot{r}(u')^2, [\dot{\bar{r}}u''' + 3\ddot{r}u'u'' + \dddot{r}(u')^3]]$

$= [\dot{\bar{r}}u', \dot{\bar{r}}u'', \dot{\bar{r}}u'''] + [\ddot{\bar{r}}u', \dot{r}u'', 3\ddot{r}u'u'']$

$+[\dot{r}u', \dot{\vec{r}}u'', \dddot{r}(u')^3] + [\dot{\bar{r}}u', \dot{r}(u')^2, \dot{r}u''']$

$+[\dot{r}u', \ddot{\vec{r}}(u')^2, 3\ddot{r}u'u''] + [\dot{r}u', \ddot{r}(u')^2\ddot{r}(u')^3]$

$= 0 + 0 + 0 + 0 + 0 + (u)^b[\dot{r}, \ddot{r}, \dddot{r}]$

$[\bar{r}', \bar{r}'', \bar{r}'''] = (u')^6[\dot{\vec{r}}, \ddot{r}, \dddot{r}]$

$k^2\tau = (u')^6[\dot{\bar{r}}, \ddot{r}, \dddot{r}]$ …………..(4)

From our assumption the curve is plane curve $\Rightarrow \tau = 0$

$\therefore$ (4) $\Rightarrow 0 = (u')^6[[\dot{r}, \ddot{r}, \dddot{r}\ ]$

$\Rightarrow [\dot{\vec{r}}, \ddot{r}, \dddot{r}] = 0$

Sufficient part:

Assume that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$ …………..(5)

To prove that: The curve is a plane curve.

(i.e.) To prove that $\tau = 0$.

Suppose $\tau \neq 0$

from (4) & (3) we get

$k^2\tau = 0$.

$\Rightarrow k^2 = 0 \Rightarrow k = 0.$

$\Rightarrow$ A curve is a straight line

This is contradiction to our assumption.

$\therefore \tau = 0.$

$\Rightarrow$ The curve is a plane curve.

Example 6:

Calculate curvature & torsion of the cubic curve given by $\bar{r} = (u, u^2, u^3)$

Solution:

Curvature $= \kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$ ………………………….. (1)

Torsion $= \tau = \frac{[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]}{|\dot{r} \times \ddot{r}|^2}$ or $\frac{[\bar{r}', \bar{r}'', \bar{r}''']}{\kappa^2}$ …………..(2)

To find: $\dot{r}, \ddot{r}$ & $\dddot{\bar{r}}$

given, $\bar{r} = (\mathrm{u}, u^2, u^3)$

Diff w.r.to ' u ', $\dot{\bar{r}} = (1, 2u, 3u^2)$, $\ddot{\bar{r}} = (0, 2u, 6u)$ and $\bar{r} = (0,0,6)$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} i & j & k \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$

$= i[12u^2 - 6u^2] - j[6u - 0] + k[2 - 0]$

$= 6u^2 i - 6uj + 2k$

$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \sqrt{(36u^4 + 6u^2 + 4)}$

$= \sqrt{4}\sqrt{(9u^4 + 9u^2 + 1)}$

$\Rightarrow |\dot{\bar{r}} \times \ddot{\bar{r}}| = 2\sqrt{9u^4 + 9u^2 + 1}$ ……… (3)

$\Rightarrow |\dot{\bar{r}} \times \ddot{\bar{r}}|^2 = 4(9u^4 + qu^2 + 1)$ …………… (4)

$$[\dot{\bar{r}}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix}$$

$= 1(12 - 0) - 2u(0 - 0) + 3u^2(0 - 0)$

$[\dot{\bar{r}}, \ddot{r}, \dddot{r}] = 12,$ ………. (5)

$\therefore$ Sub (3) in (1)

$$\text{curvature } = k = \frac{|\dot{r} \times \ddot{\bar{r}}|}{|\dot{r}|^3}$$

$$= \frac{2\sqrt{9u^4 + 9u^2 + 1}}{\left(\sqrt{1 + 4u^2 + 9u^4}\right)^3} = \frac{2(9u^4 + 9u^2 + 1)^{1/2}}{(1 + 4u^2 + 9u^4)^{3/2}}.$$

Sub (4),(5) in (2)

Torsion = $\tau = \frac{[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$

$$= \frac{12}{4(9u^4+9u^2+1)} = \frac{3}{9u^4+9u^2+1}$$

(OR)

Given $\bar{r} = (u, u^2, u^3)$ ....(1)

Differentiate with respect to u,

$\frac{d\bar{r}}{du} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{du} = (1,2u, 3u^2)$

$ie)\ \dot{\bar{r}} = \bar{r}' . \dot{s} = (1,2u, 3u^2)$

$ie)\ \dot{\bar{r}} = \bar{t} . \dot{s} = (1,2u, 3u^2)$ .... (2)

Again diff w. r. to 'u'

$$\ddot{r} = \bar{t} \cdot \ddot{S} + \dot{s}\frac{d\bar{t}}{du} = (0,2,6u)$$

$$\text{(i.e)}\ddot{\bar{r}} = \bar{t}.\ddot{s} + \dot{s}^2\bar{t}' = (0,2,6u) \left[\frac{d\bar{t}}{du} = \frac{dt}{ds} \cdot \frac{ds}{du} = \bar{t}' \cdot \dot{s}\right].$$

$$\text{(i.e) } \ddot{\bar{r}} = \bar{t}\ddot{s} + \dot{s}^2 k\bar{n} = (0,2,6u) \quad \dots \dots \dots (3)$$

Taking cross product of (2) & (3),

$\dot{\bar{r}} \times \ddot{\bar{r}} = (6u^2, 6u, 2)$

Now, $(2)^2 \Rightarrow \dot{s}^2 F^2 = 1 + 4u^2 + 9u^4$

$\Rightarrow \dot{s}^2 = 1 + 4u^2 + 9u^4$

Serret - Frenet Formulae:

The relations,

(i) $\bar{t}' = \kappa\bar{n}$ (Already proves)

(ii) $\bar{n}' = \tau\bar{b} - k\bar{t}$

(iii) $\bar{b}' = -\tau\bar{n}$ are known as the Serret Frenet Formula

{(ii) **proof :**

We know that $\bar{n} = \bar{b} \times \bar{t}$

$\bar{n}' = \bar{b}' \times \bar{t} + \bar{b} \times \bar{t}$
$= -\tau\bar{n} \times \bar{t} + \bar{b} \times k\bar{n}$
$= -\tau(\bar{h} \times \bar{t}) + k(\bar{b} \times \bar{h})$
$= -\tau(-\bar{b}) + k_1(-\bar{t})$
$= -k\bar{t} + \tau\bar{b} = \tau\bar{b} - k\bar{t}\}$

Theorem 6: [Serret-Frenet Formulae]

Prove that the behavior of a curve in the neighborhood of one of its pts may be investigated by means of relations $\bar{t}' = \kappa\bar{n}, \bar{n}' = \tau\bar{b} - k\bar{t}$ & $\bar{b}' = -\tau\bar{n}$.

Proof:

At a point ' p ' on the curve, Let axis O_x, O_y, O_z be taken along $\bar{t}, \bar{n}$ and $\bar{b}$

Let x, y, z be the co-ordinates x of a neighboring point Q of the curve relative to these axis.

If the curve is of class $\geqslant 4$.

If 's' denotes the small are length PQ. then using Taylorss theorem,

$\bar{r}(s) = \bar{r}(0) + \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \frac{s^3}{3!}\bar{r}'''(0) + \frac{s^4}{4!}r^{(\mathrm{iv})} + \mathrm{o}(\mathrm{s})$ as $\mathrm{s} \to 0$ ....(1)

given relations

$\bar{t}' = \kappa\bar{n}, \bar{n}' = \tau\bar{b} - k\bar{t}$ & $\bar{b}' = -\tau\bar{n}$. .... (2)

Here, $\bar{r}(0)$& $\bar{r}(s)$ respectively denote the position

vector of the two pts P & Q.

Let $\bar{r}(s) = (x, y, z)$ & $\bar{r}(0) = (0,0,0)$

We know that $\bar{r}'(0) = \bar{t}$

$\Rightarrow \bar{r}''(0) = \bar{t}' = \kappa\bar{n}$
$=> \bar{r}'''(0) = \kappa\bar{n}' + \kappa'^{\bar{n}}$
$= \kappa\tau\bar{b} - \kappa^2\bar{t} + \kappa'^{\bar{n}}$

$\bar{r}^{iv}(0) = -3\kappa\kappa' + [2\kappa'\tau + \kappa\tau']\bar{b} + [\kappa" - \kappa\tau^2 - \kappa^3]\,\bar{n}$

Therefore (1) $\Rightarrow$

$$(X, Y, Z) = s\bar{t} + \frac{s^2}{2!}\kappa\bar{n} + \frac{s^3}{\mathbf{3}!}\left(\kappa\tau\bar{b} - \kappa^2\bar{t} + \kappa'^{\bar{n}}\right) + \frac{s^4}{4!}(-3\kappa\kappa'^{\bar{t}} + \bar{b}(2\kappa'\tau + \kappa\tau)$$
$$+\bar{n}(\kappa" - \kappa^3 - \kappa\tau^2)$$

equating the respective co-eft of $\bar{t}, \bar{n}, \bar{b}$.

$$X = s - \frac{k^2s^3}{6} - 3kk'\frac{s^4}{24} + o(s)$$

$$Y = \frac{s^2}{2}k + \frac{k's^3}{6} + \frac{s^4}{24}[k'' - k^3 - k\tau'] + o(s)$$

$Z = \kappa\tau\frac{s^3}{6} + \frac{1}{24}(2k'\tau + k\tau')s^4 + o(s)$

The co-eff. being evaluated at P.

It follows that as a first order approximation the chord PQ in along the tangent.

(ie) $\bar{r}(s) = \bar{r}(0) + s(\bar{t})$

$\bar{r}(s) - \bar{r}(0) = s\bar{t}$

$\overline{OQ} - \overline{OP} = S\bar{t}$

(i.e) The projection on the principal normal is a magnitude of the second order and its projection on the binormal is of the third order.

From eqn. (2) ⇒.

$$\frac{2Y}{X^2} = \frac{\kappa s^2 + \frac{\kappa'}{3}s^3 + \cdots.}{s^2\left(1 - \frac{k^2 s^2}{t} - \cdots\right)} = \frac{\kappa + \frac{\kappa'}{3}s + \cdots}{\left(1 - \frac{\kappa^2 s^2}{6} - \cdots\right)^2} \quad \sim \kappa \text{ as } s \to 0$$

Similarly, $\frac{3Z}{XY} \cong \frac{\kappa\tau\frac{s^2}{2}}{s\left(1 - \frac{\kappa^2 s^2}{6} - \cdots\right)\left(\frac{\kappa s^2}{2}(1 + \cdots)\right)} \quad \sim \tau$

This is similar to Newton's formula for curvature.

To find approximate length of the chord pQ

$$\left(s - \frac{\kappa^2 s^3}{6} + \cdots\right)^2 + \left(\frac{k}{2}s^2 + \cdots\right)^2 + \left(\frac{k\tau}{6}s^3 + \cdots\right)^2$$

$$X^2 + Y^2 + Z^2 = s^2 - \frac{2sk^2 s^3}{6} + \frac{k^2 s^4}{4} + \cdots = s^2[1 - \frac{1}{12}\kappa^2 s^2]$$

∴ length of $PQ \sim S\left[1 - \frac{k^2 s^2}{12}\right]$

∴ length of chord $PQ - S \sim -\frac{K^2 s^2}{24}$

When $k \neq 0$

The are length PQ differs from the chord PQ

by terms of the third order in ' s '.

Rectifying plane:

The plane determined by the tangent and binormal at ' P ' as the rectifying plane.

Example 7:

Show that the projection of the curve near P on the Osculating plane is approximately the curve $z = 0$,

$y = \frac{1}{2}kx^2$, its projection on the rectifying plane is approximately $y = 0, z = \frac{1}{6}k\tau X^3$ and its projection on the normal plane is approximately $x = 0, z^2 = \frac{2}{9}\left(\frac{\tau^2}{\kappa}\right) Y^3$

Proof:

We know that, the coordinate x, y, z of a near point to p are given by,

$$X = s - \frac{k^2 s^3}{6} - 3kk'\frac{s^4}{24} + o(s)$$

$$Y = \frac{s^2}{2}k + \frac{k' s^3}{6} + \frac{s^4}{24}[k'' - k^3 - k\tau'] + o(s)$$

$$Z = \kappa\tau\frac{s^3}{6} + \frac{1}{24}(2k'\tau + k\tau')s^4 + o(s)$$

The eqn of the osculating plane is $z = 0, x = s, y = \frac{\kappa}{2}s^2$ nearly,

(i.e.) $y = \frac{k}{2}x^2$

The eau, of the rectifying plane is $y = 0$.

$$y = 0, x = s, z = \frac{k\tau s^3}{6} \Rightarrow z = \frac{k\tau}{6}x^3.$$

$$\frac{y^3}{z^2} = \frac{(\frac{k}{2})^3 s^6}{\left(\frac{k\tau}{6}\right)^2 s^6} = \frac{\frac{k^3}{8}}{\frac{k^2\tau^2}{36}} = \frac{9}{2}\frac{k}{\tau^2}$$

(ie) $\frac{y^3}{z^2} = \frac{9}{2}\frac{k}{\tau^2} \Rightarrow z^2 = \frac{2}{9}\frac{\tau^2 y^3}{k}$

in the equation of the normal plane.

Example 8:

Show that the length of the common perpendicular 'd' of the tangent at two near points distance 's' apart in approximately given by $d = \frac{k\tau s^3}{12}$.

Proof :

Let P, Q have parameters O and S repectively.

The unit tangent vectors at P and Q are $\bar{r}'(0), \bar{r}'(S)$

$\therefore$ The unit vector of the common perpendicular in along $\bar{r}'(s) \times \bar{r}'(0)$

The projection of the vector $[\bar{r}(s) - \bar{r}(0)]$ in this direction $= d$

$\therefore d = \overline{PQ} = \frac{[\bar{r}(s) - \bar{r}(0), \bar{r}'(s), \bar{r}'(0)]}{|[\bar{r}'(s) \times \bar{r}'(0)]|} \ldots \ldots \ldots \ldots (1)$

We know that (by Taylor's Theorem),

$\bar{r}(s) = \bar{r}(0) + \frac{s}{1!}\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \frac{s^3}{3!}\bar{r}'''(0) + 0(s)\ as\ s \to \infty \ldots \ldots \ldots \ldots (2)$

We know that, $\bar{r}'(0) = \bar{t}$

$\Rightarrow \bar{r}''(0) = \bar{t}'$

$\bar{r}'''(0) = -\kappa^2\bar{t} + \kappa'\bar{r} + \kappa\tau\bar{b}$

$= \kappa'(\tau\bar{b} - \kappa\bar{t}) + \kappa'\bar{n}$

$= \kappa(\tau\bar{b} - \kappa\bar{t}) + \kappa'\bar{n}$

$\because$ (1) $\Rightarrow$

$$\therefore \bar{r}(s) - \bar{r}(0) = \frac{s}{1}\bar{t} + \frac{s^2}{2}\kappa\bar{n} + \frac{s^3}{6}\left(-\kappa^2\bar{t} + \kappa'\bar{\tilde{n}} + \kappa\tau\bar{b}\right) \text{ ....(A)}$$

$$\Rightarrow \bar{r}(s) - \bar{r}(0) = \bar{t}\left[\frac{s}{1} - \frac{s^2k^2}{6}\right] + \bar{n}\left[\frac{\kappa s^2}{2} + \frac{\kappa' s^3}{6}\right] + \bar{b}\left[\frac{k\tau s^3}{6}\right]$$

Differentiate (A) with respect to s,

$$\Rightarrow \bar{r}'(s) = \bar{t}\left[1 - \frac{2s\kappa^2}{6}\right] + \bar{n}\left[\frac{\kappa}{2}\cdot 2s + \frac{3\kappa' s^2}{6}\right] + \bar{b}\left[\frac{3\kappa\tau s^2}{6}\right] \text{ ....}(B)$$

$\Rightarrow \bar{r}'(0) = \bar{t}[1-0] + \bar{n}[0] + \bar{b}[0]$

$\Rightarrow \bar{r}'(0) = \bar{t}$ ....(C)

$$\therefore \bar{r}'(s) \times \bar{r}'(0) = \left[\bar{t}\left(1 - \frac{\kappa^2 s^2}{2}\right) + \bar{n}\left(\kappa s + \frac{\kappa' s^2}{2}\right) + \bar{b}\left(\frac{s^2}{2}k\tau\right)\right] \times \bar{t}$$

$$= 0 - \bar{b}\left[\kappa + \frac{\kappa' s^2}{2}\right] + \bar{n}\left[\frac{s^2}{2}\kappa\tau\right] \quad \begin{matrix}[\bar{n}\times\bar{t} = -\bar{b} \\ \bar{b}\times\bar{t} = \bar{n}]\end{matrix}$$

$$= -\bar{b}\left[s\kappa + \kappa'\frac{s^2}{2}\right] + \bar{n}\left[\frac{s^2}{2}\kappa\tau\right]$$

$$\Rightarrow |\bar{r}'(s) \times \bar{r}'(0)| = \left|-\bar{b}\left[s\kappa + \kappa'\frac{s^2}{2}\right] + \bar{n}\left[\frac{s^2}{2}\kappa\tau\right]\right|$$

$$= \sqrt{\left[s\kappa + \kappa'\frac{s^2}{2}\right]^2 + \frac{s^4}{4}\kappa^2\tau^2}$$

$$= \left[s^2\kappa^2 + \kappa'^2\frac{s^4}{4} + 2s\kappa\kappa'\frac{s^2}{2} + \frac{s^4}{4}\kappa^2\tau^2\right]^{1/2}$$

$$= \left[s^2\kappa^2 + \kappa'\frac{s^4}{4} + \kappa\kappa' s^3 + \frac{s^4}{4}\kappa^2\tau^2\right]^{1/2}$$

$= [s^2\kappa^2 + s^3\kappa\kappa']^{1/2}$ [omit high powers]

$= s[\kappa^2 + s\kappa\kappa']^{1/2}$

$$= s\kappa\left[1 + \frac{s}{\kappa}\kappa'\right]^{1/2}$$

(i.e.) $|\bar{r}'(s) \times \bar{r}'(0)| = s\kappa\left[1 + \frac{s}{\kappa}\kappa'\right]^{1/2}$ (approximation)

$\sim s\,\kappa\left[1 + \frac{s}{2n}n'\right]$ ....(3) $[\because (1+x)^n = 1 + nc_1x + hc_2x^2 + \cdots]$

From (A), (B) & (C)

$$[\bar{r}(s)-\bar{r}(0), \bar{r}'(s), \bar{r}'(0)] = \begin{vmatrix} s-\frac{s^2\kappa^2}{6} & \kappa\frac{s^2}{2}+\frac{\kappa' s^3}{6} & \frac{\kappa\tau s^3}{6} \\ 1-\frac{s k^2}{3} & \kappa s+\frac{\kappa' s^2}{2} & \frac{\kappa\tau s^2}{2} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \left[s-\frac{s^2\kappa^2}{6}\right][0-0] - \left[\frac{\kappa s^2}{2}+\frac{\kappa\prime s^3}{6}\right]\left[0-\frac{\kappa\tau s^2}{2}\right] + \left[\frac{\kappa\tau s^3}{6}\right]\left[0-\kappa s-\frac{\kappa' s^2}{2}\right]$$

$$= 0+\frac{\kappa^2\tau s^4}{4}+\frac{\kappa\kappa'\tau s^5}{12}-\frac{\kappa^2\tau s^4}{6}-\frac{\kappa\kappa'\tau s^5}{12}$$

$$= \frac{\kappa^2\tau^4}{2}\left[\frac{1}{2}-\frac{1}{3}\right]$$

$$= \frac{\kappa^2\tau s^4}{12} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Sub (3) & (4) in (1),

$$\therefore (1) \Rightarrow d = \frac{\frac{\kappa^2\tau s^4}{12}}{s\kappa\left[1+\frac{s}{\kappa}\kappa'\right]^{1/2}}$$

$$= \frac{\tau s^3\kappa}{12\left[1+\frac{s}{2\kappa}\kappa'\right]} \text{ [omit high powers]}$$

$$= \frac{\kappa\tau s^3}{12}\left[1+\frac{s}{2\kappa}\kappa'\right]^{-1}$$

$$= \frac{\kappa\tau s^3}{12}\left[1-\frac{s}{2\kappa}\kappa'\right]$$

$$= \frac{\kappa\tau s^3}{12}-\frac{\tau s^4\kappa'\kappa}{24\kappa}$$

$$d \sim \frac{s^3\kappa\tau}{12} \text{ (nearly)}$$

1.5. Curvature and torsion of a curve given as the intersection of two surfaces:

Theorem 1:

If a curve is given as the intersection of two surfaces, $f(x,y,z)=0, g(x,y,z)=0$ and if a set of parametric equations for the curve cannot readily obtained, then explain the method of the curvature and torsion of the curve.

(or)

Explain the method calculating the curvature and torsion of the curve. Given as the intersection of two surfaces.

Solution:

Given a curve is the intersection of two surfaces $f(x,y,z)=0, g(x,y,z)=0$. & also $given$ a set of of parametric eqn's for the curve cannot readily to obtained.

Let the curve of intersection be represented by the equation $\bar{r}=\bar{r}(u)$

Let the two surfaces be given by,

$f(\bar{r}) = 0, g(\bar{r}) = 0$

Now, The unit tangent vector to the curve in orthogonal to the normal of both surfaces. Thus,

if $\nabla \bar{f} = \left[\frac{\partial \bar{f}}{\partial x}, \frac{\partial \bar{f}}{\partial y}, \frac{\partial \bar{f}}{\partial z}\right]$

& $\nabla \bar{g} = \left[\frac{\partial \bar{g}}{\partial x}, \frac{\partial \bar{g}}{\partial y}, \frac{\partial \bar{g}}{\partial z}\right]$

$\therefore$ $\bar{t}$ is parallel to $\nabla \bar{f} \times \nabla \bar{g}$

Let $\nabla \bar{f} \times \nabla \bar{g} = \bar{h}$

$\therefore \lambda \bar{t} = \nabla \bar{f} \times \nabla \bar{g} = \bar{h}$

$\lambda \bar{r}' = \bar{h}$, for some $\lambda \longrightarrow (A)$

$\lambda \frac{d\bar{r}}{ds} = \bar{h} \longrightarrow (1)$

then $\lambda x' = h_1, \lambda y' = h_2, \lambda \bar{z}' = h_3$

& $\lambda \frac{d}{ds} = \left[h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z}\right] = \Delta$ (say) ………..(2)

$\left[\because \lambda \frac{d\bar{r}}{ds} = \bar{h}\right.$

Substituting in (1),

$\Rightarrow \lambda \left[\frac{d\bar{r}}{dx} \cdot \frac{dx}{ds} + \frac{d\bar{r}}{dy} \cdot \frac{dy}{ds} + \frac{d\bar{r}}{dz} \cdot \frac{dz}{ds}\right] = (h_1, h_2, h_3).$

$\Rightarrow \lambda \left[\frac{d\bar{r}}{dx} \cdot x' + \frac{d\bar{r}}{dy} \cdot y' + \frac{d\bar{r}}{dz} \cdot z'\right] = (h_1, h_2, h_3)$

Comparing the coefficients,

we get, $\lambda x' = h_1$

$\lambda y' = h_2$

$\lambda z' = h_3$

$\therefore \lambda \frac{d\bar{r}}{ds} = \left[\lambda x' \frac{d\bar{r}}{dx} + \lambda y' \frac{d\bar{r}}{dy} + \lambda z' \frac{d\bar{r}}{dz}\right]$

$\Rightarrow \lambda \frac{d\bar{r}}{ds} = \left[\lambda x' \frac{d}{dx} + \lambda y' \frac{d}{dy} + \lambda z' \frac{d}{dz}\right] \bar{r}$

$\Rightarrow \lambda \frac{d}{ds} = \lambda x' \frac{d}{dx} + \lambda y' \frac{d}{dy} + \lambda z' \frac{d}{dz} = \Delta$ (let)

$\left. = h_1 \frac{d}{dx} + h_2 \frac{d}{dy} + h_3 \frac{d}{dz} = \Delta \right]$

$\therefore (1) \Rightarrow \lambda \frac{d\bar{r}}{ds} = \bar{h}$

$$\Rightarrow \Delta\bar{r} = \bar{h} \quad \ldots\ldots\ldots\ldots..(3)$$

$We\ know\ that\ \lambda\bar{t} = \bar{h}\ \ldots\ldots\ldots\ldots.(4)$

$$\Rightarrow \lambda\bar{t}\cdot\lambda\bar{t} = \bar{h}\cdot\bar{h}$$
$$\Rightarrow \lambda^2(\bar{t}\cdot\bar{t}) = \bar{h}^2$$
$$\Rightarrow \lambda^2(1) = h^2 \Rightarrow \lambda^2 = \bar{h}^2 \quad \ldots\ldots\ldots..(5)$$

From (4) $\Rightarrow \lambda\bar{t} = \bar{h} \rightarrow \Delta(\lambda\bar{t}) = \Delta\bar{h}$

$$\Rightarrow \lambda\frac{d}{ds}(\lambda\bar{t}) = \Delta\bar{h}$$

$$\Rightarrow \lambda[\lambda'\bar{t} + \lambda\bar{t}'] = \Delta\bar{h} \quad [\because (2)]$$

$$\Rightarrow \lambda[\lambda\bar{t} + \lambda\kappa\bar{n}] = \Delta\bar{h}$$

$$\Rightarrow \lambda\lambda'\bar{t} + \lambda^2\kappa\bar{n} = \Delta\bar{h} \quad \ldots\ldots\ldots\ldots..(6)$$

Taking cross product of ' λt ' with (6),

$\therefore$ (5) $\Rightarrow$

$$\lambda\bar{t}\times(\lambda\lambda'\bar{t} + \lambda^2\kappa\bar{n}) = \lambda\vec{\bar{t}}\times\Delta\bar{h}$$

$$0 + \lambda^3\kappa(\bar{t}\times\bar{n}) = \bar{h}\times\Delta\bar{h}$$

$$\Rightarrow \lambda^3\kappa\bar{b} = \bar{h} + \Delta\bar{h} = \bar{k}(\text{ say }) \ldots\ldots\ldots\ldots(7)$$

Equation (7) gives curvature $\bar{k}$

Taking dot product with itself in equation (7) on both sides,

equation (7) on both sides,

$$(\lambda^3\kappa\bar{b})\cdot(\lambda^3\kappa\bar{b}) = \bar{\kappa}\cdot\bar{\kappa}$$

$$\lambda^6\kappa^2(1) = \kappa^2$$

$$\Rightarrow \kappa^2 = \frac{\bar{k}^2}{\lambda^6}$$

$\therefore$ The curvature is $\kappa = \frac{\bar{k}}{\lambda^3}$ $\ldots\ldots\ldots\ldots(8)$

Apply the operator ' Δ ' on equation (7),

$$\Delta(\lambda^3\kappa\bar{b}) = \Delta\bar{k}$$

$$\Rightarrow \lambda\frac{d}{ds}(\lambda^3\kappa\bar{b}) = \Delta\bar{k}$$

$$\Rightarrow \lambda\left[(\lambda^3\kappa)'\bar{b} + (\lambda^3\kappa)\bar{b}'\right] = \Delta\bar{k}$$

$$\Rightarrow \lambda[\lambda^3\kappa]'\bar{b} + \lambda^4\kappa(-\tau\bar{n}) = \Delta\bar{k}\ [\bar{b} = -\tau\bar{n}]$$

$$\Rightarrow [\lambda(\lambda^3\kappa)'\bar{b}] - \tau\lambda^4\kappa\bar{n} = \Delta\bar{k} \ldots\ldots\ldots\ldots.(9)$$

Taking dot product of (6) with (9),

$[\lambda\lambda'\bar{t}+\lambda^2\kappa\bar{n}]\cdot[[\lambda(\lambda^3\kappa)'\bar{b}]-\lambda^2\kappa\tau\bar{n}]=\bar{h}\cdot\bar{k}$

$0-0+0-\lambda^6\kappa^2\tau(\bar{n}.\bar{n})=\Delta\bar{h}.\Delta\bar{k}$

$-\lambda^6\kappa^2\tau(\bar{n}.\bar{n})=\Delta\bar{h}.\Delta\bar{k}$

Substitute the values ' λ ' from (3) and κ from equation (10)

we get τ.

Example 1:

Obtain the curvature and torsion of the curve of intersection of the two quadric surfaces

$ax^2+by^2+cz^2=1, a'x^2+b'y^2+c'z^2=1.$

Solution:

Let $f=\frac{1}{2}(ax^2+by^2+cz^2-1)$ and $g=\frac{1}{2}(a'x^2+b'y^2+c'z^2-1)$

So, $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=(ax,by,cz)$

Similarly, $\nabla\bar{g}=(a'x,b'y,c'z)$

$$\nabla\bar{f}\times\nabla\bar{g}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \bar{k}\\ ax & by & cz\\ a'x & b'y & c'z\end{vmatrix}$$

$=\vec{\imath}(bc'-cb')yz+j(a'c-c'a)xz+\bar{k}(ab'-a'b)xy.$

$=(Ayz,Bxz,Cxy)$

$\nabla\bar{f}\times\nabla\bar{g}=xyz\left(\frac{A}{x},\frac{B}{y},\frac{C}{z}\right)$

$\Rightarrow\lambda_1\bar{r}'=xyz\left(\frac{A}{x},\frac{B}{y},\frac{C}{z}\right).$ $\quad[\because\nabla\bar{f}\times\nabla\bar{g}=\lambda\bar{r}=\lambda\bar{t}]$

Since, $\bar{r}'=(x',y',z')$ is parallel to $\nabla\bar{f}\times\nabla\bar{g}$

$\therefore$ we choose, λ_1 such that,

$\lambda_1(x',y',z')=xyz\left(\frac{A}{x},\frac{B}{y},\frac{C}{z}\right)$

$\Rightarrow\frac{\lambda_1}{xyz}(x',y',z')=\left(\frac{A}{x},\frac{B}{y},\frac{C}{z}\right)$

$\Rightarrow\lambda(x',y',z')=\left(\frac{A}{x},\frac{B}{y},\frac{c}{z}\right)$ $\quad\left[\because \text{ put } \frac{\lambda_1}{xyz}=\lambda\right]$

$\Rightarrow\lambda\bar{t}=\left(\frac{A}{x},\frac{B}{y},\frac{C}{z}\right)$ ... . (1). $\quad[\because\ \bar{t}=\bar{r}']$

operate Δ on (1)

(1) $\Rightarrow\Delta\lambda\bar{t}=\Delta\left(\frac{A}{x},\frac{B}{y},\frac{c}{z}\right)$

$$\lambda \frac{d}{ds}[\lambda \bar{t}] = \lambda \frac{d}{ds}\left(\frac{A}{x}, \frac{B}{y}, \frac{c}{z}\right) \qquad \left[\because \Delta = \lambda \frac{d}{ds}\right]$$

$$\lambda[\lambda' \bar{t} + \bar{\lambda} t'] = \left[h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z}\right]\left(\frac{A}{x}, \frac{B}{y}, \frac{c}{z}\right) \qquad \left[\because \lambda \frac{d}{ds} = h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z}\right]$$

$$\lambda' \lambda' \bar{t} + \lambda^2 \kappa \bar{n} = \left[\left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{y}\right)\right]$$

$$\left[\because \lambda \bar{t} = \bar{h} \Rightarrow \left(\frac{A}{x}, \frac{B}{y}, \frac{c}{z}\right) = (h_1, h_2, h_3)\right]$$

$$\Rightarrow h_1 = \frac{A}{x}, h_2 = \frac{B}{y}, h_3 = \frac{c}{z}$$

$$= \left[\left[\frac{A}{x}\frac{\partial}{\partial x}\left(\frac{A}{x}\right) + \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{A}{x}\right) + \frac{c}{z}\frac{\partial}{\partial z}\left(\frac{A}{x}\right)\right], \left[\frac{A}{x}\frac{\partial}{\partial x}\left(\frac{B}{y}\right) + \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{B}{y}\right) + \frac{C}{z}\frac{\partial}{\partial z}\left(\frac{B}{y}\right)\right], \left[\frac{A}{x}\frac{\partial}{\partial x}\left(\frac{c}{z}\right) + \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{c}{z}\right) + \frac{c}{z}\frac{\partial}{\partial z}\left(\frac{c}{z}\right)\right]\right]$$

$$= \left[\left(\frac{A}{x}\left(-\frac{A}{x^2}\right) + 0 + 0\right), \left(0 + \frac{B}{y}\left(-\frac{B}{y^2}\right) + 0\right), \left(0 + 0 + \frac{C}{z}\left(\frac{-C}{z^2}\right)\right)\right]$$

$$\therefore \lambda' \lambda' \bar{t} + \lambda^2 \kappa \bar{n} = \left[-\frac{A^2}{x^3}, -\frac{B^2}{y^3}, -\frac{C^2}{z^3}\right] \quad \ldots \ldots \ldots \quad (2)$$

Find (1) cross (2);

$$[\lambda \bar{t}] \times [\lambda \lambda' \bar{t} + \lambda^2 \kappa \bar{n}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A/x & B/y & c/z \\ -A^2/x^3 & -B^2/y^3 & -c^2/z^3 \end{vmatrix}$$

$$\lambda \bar{t} \times \lambda^2 \bar{t} + \lambda \bar{t} \times \lambda^2 \kappa \bar{n} = \vec{i}\left[-\frac{BC^2}{yz^3} + \frac{CB^2}{y^3 z}\right] + j\left[-\frac{AC^2}{xz^3} + \frac{CA^2}{x^3 z}\right] + \vec{k}\left[-\frac{AB^2}{xy^3} + \frac{BA^2}{x^3 y}\right]$$

$$\lambda^3 \kappa \bar{b} = i\left[\frac{-BC^2 y^2 + CB^2 z^2}{y^3 z^3}\right] + j\left[\frac{x^2 AC^2 - z^2 A^2 c}{x^3 z^3}\right] + \bar{k}\left[\frac{-AB^2 x^2 + BA^2 y^2}{x^3 y^3}\right]$$

$$\lambda^3 \kappa \bar{b} = \left[\left(\frac{-BC^2 y^2 + CB^2 z^2}{y^3 z^3}\right), \left(\frac{x^2 Ac^2 - z^2 A^2 c}{x^3 z^3}\right), \left(\frac{-AB^2 x^2 + BA^2 y^2}{x^3 y^3}\right)\right] \ldots \ldots . (3)$$

Let $-BC^2 y^2 + CB^2 z^2 = BC[-Cy^2 + Bz^2]$.

$= BC[(-ab' + ba')y^2 - (ca' + ac')z^2]$

$= BC[-ab'y^2 + ba'y^2 - ca'z^2 - ac'z^2]$

$= BC[-a(b'y^2 + c'z^2) + a'(by^2 - cz^2)]$

$= BC\big(-a(1 - a'x^2) + aa'(1 - ax^2)\big)$

$[\because a'x^2 + b'y^2 + c'z^2 = 1]$

$\& ax^2 + by^2 + cz^2 = 1]$

$= BC(-a + aa'x^2 + a' - aa'x^2)$

$= BC(a' - a)$ .... (4)

Similarly $-A^2cz^2 + Ac^2x^2 = AC(b' - b)$ .... (5)

$AB^2x^2 + A^2By = AB(c' - c)$ .... (6)

$$(3) \Rightarrow \lambda^3\kappa\bar{b} = \left[\frac{CB(a'-a)}{y^3z^3}, \frac{AC(b'-b)}{x^3z^3}, \frac{BA(c'-c)}{x^3y^3}\right]$$

$$= \frac{ABC}{x^3y^3z^3}\left[\frac{(a'-a)x^3}{A}, \frac{(b'-b)y^3}{B}, \frac{(c'-c)z^3}{c}\right] \ldots\ldots\ldots\ldots (7)$$

Taking (.) product of (7) with itself,

$$\Rightarrow (\lambda^3\kappa\bar{b}) \cdot (\lambda^3\kappa\bar{b}) = \frac{A^2B^2c^2}{x^6y^6z^6}\sum\frac{(a'-a)^2x^6}{A^2}.$$

$$\Rightarrow \lambda^6\kappa^2(1) = \frac{A^2B^2c^2}{x^6y^6z^6}\sum\frac{(a'-a)^2x^6}{A^2} \ldots\ldots\ldots (8)$$

Taking (.) product of (1) with itself,

$$(1) \Rightarrow (\lambda\bar{t}) \cdot (\lambda\bar{t}) = \sum\frac{A^2}{x^2}$$

$$\Rightarrow \lambda^2(1) = \sum\frac{A^2}{x^2} \Rightarrow \lambda^6 = \left[\sum\left(\frac{A^2}{x^2}\right)\right]^3 \ldots (*) \text{ sub in } (8)$$

$\therefore$ (8) $\Rightarrow$

$$\left[\sum\left(\frac{A^2}{x^2}\right)\right]^3\kappa^2 = \frac{A^2B^2c^2}{x^6y^6z^6}\sum\frac{(a'-a)^2x^6}{A^2}.$$

$$\Rightarrow \kappa^2 = \frac{A^2B^2c^2}{x^6y^6z^6}\frac{\sum\frac{(a'-a)^2}{A^2}x^6}{\left[\sum\left(\frac{A^2}{x^2}\right)\right]^3} \ldots\ldots\ldots (9)$$

$$(7) \Rightarrow \lambda^3\kappa\bar{b} = \frac{ABC}{x^3y^3z^3}\left[\frac{(a'-a)x^3}{A}, \frac{(b'-b)y^3}{B}, \frac{(c'-c)z^3}{C}\right]$$

put $\lambda^3\kappa\frac{x^3y^3z^3}{ABC} = \mu$

$$\Rightarrow \mu\bar{b} = \left[\frac{(a'-a)x^3}{A}, \frac{(b'-b)y^3}{B}, \frac{(c'-c)z^3}{C}\right]$$

Taking Δ on both sides, $\left[\because \Delta = \lambda\frac{d}{ds}\right]$

$$\Rightarrow \lambda\frac{d}{ds}(\mu\bar{b}) = \lambda\frac{d}{ds}\left[\frac{(a'-a)x^3}{A}, \frac{(b'-b)y^3}{B}, \frac{(c'-c)z}{c}\right]$$

$$\Rightarrow \lambda[\mu'\bar{b} + \mu\,\bar{b}'] = \left[\frac{A}{x}\cdot\frac{\partial}{\partial x} + \frac{B}{y}\cdot\frac{\partial}{\partial y} + \frac{c}{z}\cdot\frac{\partial}{\partial z}\right].\left[\frac{(a'-a)x^3}{A}, \frac{(b'-b)y^3}{B}, \frac{(c'-c)z^3}{c}\right]$$

$$\lambda[\mu'\bar{b}-\mu\tau\bar{n}] = \left[\left(\frac{A}{x}\frac{\partial}{\partial x}+\frac{B}{y}\frac{\partial}{\partial y}+\frac{c}{z}\frac{\partial}{\partial z}\right]\left(\frac{(a'-a)x^3}{A}\right]\right.$$

$$\left[\frac{A}{x}\cdot\frac{\partial}{\partial x}+\frac{B}{y}\cdot\frac{\partial}{\partial y}+\frac{c}{z}\cdot\frac{\partial}{\partial z}\right]\left[\frac{(b'-b)y^3}{B}\right],$$

$$\left[\frac{A}{x}\cdot\frac{\partial}{\partial x}+\frac{B}{y}\cdot\frac{\partial}{\partial y}+\frac{c}{z}\cdot\frac{\partial}{\partial z}\right]\left[\frac{(c'-c)z^3}{c}\right]\Bigg]$$

$$=\left[\left[\frac{A}{x}\frac{3x^2(a'-a)}{\mathrm{A}}+0+0\right],\right.$$

$$\left[0+\frac{3y(b'-b)}{B}+0\right],$$

$$\left.\left[0+0+\frac{3Cz^2(c'-c)}{\mathrm{C}}\right]\right]$$

$$\lambda\mu'\bar{b}-\lambda\mu\tau\bar{n} = [3x(a'-a),3y(b'-b),3z(c'-c)]$$

Taking ($\cdot$) product of (2) with (10)

$$[\lambda\lambda'\bar{t}+\lambda^2\kappa\bar{n}]\cdot[\lambda\mu\bar{b}-\lambda\mu\tau\bar{n}] = \left[-\frac{A^2}{x^3},\frac{-B^2}{y^3},\frac{-c^2}{z^3}\right][3x(a'-a),3y(b'-b),3z(c'-c)]$$

$$\Rightarrow [0-0+0-\lambda^3\kappa\mu\tau] = -\sum\frac{3A^2(a'-a)}{x^2}$$

(ie) $\lambda^3\mu\kappa\tau = \sum\frac{3A^2(a'-a)}{x^2}$

$$\tau = \frac{3}{\lambda^3\mu\kappa}\sum\frac{A^2(a'-a)}{x^2}\left[\because \mu = \frac{\lambda^3\kappa x^3y^3z^3}{ABC}\right]$$

$$= \frac{3ABC}{\lambda^6\kappa\kappa\, x^3y^3z^3}\sum\frac{A^2(a'-a)}{x^2}$$

$$= \frac{3ABC}{\lambda^6\kappa^2x^3y^3z^3}\sum\frac{A^2(a'-a)}{x^2}$$

$$= \frac{3ABC}{\lambda^6x^3y^3z^3}\cdot\frac{x^6y^6z^6}{A^2B^2C^2}\left[\frac{\sum(A^2/x^2)^3}{\sum\frac{(a'-a)^2x^6}{A^2}}\right]\cdot\left[\sum\frac{A^2(a'-a)}{x^2}\right]$$

$$= \frac{3x^3y^3z^3}{\lambda^6ABC}\frac{\sum\frac{A^2(a'-a)}{x^2}}{\sum\frac{x^6(a^1-a)^2}{A^2}}[\sum(A^3/x^2)^3]$$

$$\left[\because (*) \Rightarrow \lambda^6 = \left[\Sigma\left(\frac{A^2}{x^2}\right)\right]^3\right]$$

$$\tau = \frac{3x^3y^3z^3}{ABC}\frac{1}{[\Sigma A^2/x^2]^3}\cdot\frac{\Sigma\frac{A^2(a^1-a)}{x^2}}{\Sigma\left[\frac{x^6(a'-a)^2}{A^2}\right]}[\Sigma A^2/x^2]^3$$

$$\therefore \tau = \frac{3x^3y^3z^3}{ABC}\frac{\Sigma\frac{A^2(a'-a)}{x^2}}{\Sigma\left[\frac{x^6(a'-a)^2}{A^2}\right]} \quad \ldots\ldots\ldots\ldots (11)$$

$\therefore$ equation (9) is curvature and equation (11) is Torsion.

1.6. Contact Between Curves and Surfaces:

Let γ be a curve of sufficiently high class, given by the equation $\bar{r} = \{f(u), g(u), h(u)\}$ and let S be a surface given by $F(x, y, z) = 0$. Where the function f has a sufficiently high class then $'\gamma'$ and 's' are said to be 'n' – point contact if $F'(u_0) = F''(u_0) = \cdots = F^{(n-1)}(u_0) = 0$ with $F^{(n)}(u_0) \neq 0$.

Proof:

Given γ be a curve of sufficiently high class and equation of the curve,

$\bar{r} = \{f(u), g(u), h(u)\}$ ………….. (1)

Also given 's' be a surface given by $F(x, y, z) = 0$ … … … … …. (2)

where the function F has a sufficiently high class then, the parameters of points of γ.

Which also lie on S are zero's of the function $F(u) = F\{f(u), g(u), h(u)]$.

If u_0 is such zero. Then the function F(u) may be expressed by Taylor's theorem in the form,

$F(u) = \varepsilon F'(u_0) + \frac{\varepsilon^2}{2!}F''(u_0) + \cdots + \frac{\varepsilon^n}{n!}F^{(n)}(u_0) + 0(\varepsilon^{n+1})$ … … … … .. (3) as $u \to u_0$

Where $\varepsilon = u - u_0$

If $F'(u_0) \neq 0$, then $'u_0'$ is a simple zero of F(u) and in this case γ and in this case $'\gamma'$ and 's' have a simple intersection of $\bar{r}(u_0)$.

If $F'(u_0) = 0$, but $F'(u_0) \neq 0$, then F(u) is of the second order of ε.

$'u_0'$ is a double zero of F(u) and γ and s have two point contact.

If $F'(u_0) = F''(u_0) = 0$ but $F'''(u_0) \neq 0$, then $'u_0'$ is a triple zero of F(u) and γ and s have three point contact.

In general, if $F'(u_0) = F''(u_0) = F'''(u_0) = \cdots = F^{(n-1)}(u_0) = 0$ but $F^{(n)}(u_0) \neq 0$, then γ nad s are said to have n-point contact of $\bar{r}(u_0)$.

Example 1:

Show that the osculating plane at P has in general three point contact with the curve at 'p' with 's' as parameter measured from p(s=0).

Proof:

We have $F(s) = [\bar{r}(s) - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)]$

Let $\bar{r} = \bar{r}(s)$ be the equation of the curve with 's' as a parameter.

Then the equation of the osculating plane $[\bar{r}(s) - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0$.

We assume that the point with $u = 0$ as the parameter as the point of contact .

$\therefore F(s) = [R - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)]$

Where $\bar{r}(s) - \bar{r}(0) = s\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \frac{s^3}{3!}\bar{r}'''(0) + o(s^3)$

$$\therefore F(s) = [s\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \frac{s^3}{3!}\bar{r}'''(0) + o(s^3), \bar{r}'(0), \bar{r}''(0)]$$

$= [s\bar{r}'(0), \bar{r}'(0), \bar{r}''(0)] + \left[\frac{s^2}{2!}\bar{r}''(0), \bar{r}'(0), \bar{r}''(0)\right] + \left[\frac{s^3}{3!}\bar{r}''(0), \bar{r}'(0), \bar{r}''(0)\right] + o(s^3)$ as $s \to 0$

$= 0 + 0 + \left[\frac{s^3}{6}\bar{r}''(0), \bar{r}'(0), \bar{r}''(0)\right] + o(s^3)$ as $s \to 0$

$= \frac{s^3}{6}[\bar{r}''(0), \bar{r}'(0), \bar{r}''(0)] + o(s^3)$ as $s \to 0$

$[\because \bar{r}'(0) = \bar{t}, \bar{r}''(0) = \bar{t}' = \kappa\bar{n}$

$\bar{r}'''(0) = \kappa\bar{n}' + \kappa'\bar{n}$

$= \kappa\tau\bar{b} - \kappa^2\bar{t} + \kappa'\bar{n}]$

$= \frac{s^3}{6}\left[\kappa\tau\bar{b} - \kappa^2\bar{t} + \kappa'\bar{n}, \bar{t}, \kappa\bar{n}\right] + o(s^3)$ as $s \to 0$

$= \frac{s^3}{6}\left[[\kappa\tau\bar{b}, \bar{t}, \kappa\bar{n}] - [\kappa^2\bar{t}, \bar{t}, \kappa\bar{n}] + [\kappa'\bar{n}, \bar{t}, \kappa\bar{n}]\right] + o(s^3)$ as $s \to 0$

$= \frac{s^3}{6}\left[[\kappa\tau\bar{b}, \bar{t}, \kappa\bar{n}] - 0 + 0\right] + o(s^3)$ as $s \to 0$

$= \frac{s^3}{6}\kappa^2\tau[\bar{b}, \bar{t}, \bar{n}] + o(s^3)$ as $s \to 0$

$= \frac{s^3}{6}\kappa^2\tau \neq 0$ as $s \to 0$

$\because F'(s) = 0 = F''(s)$ as $s \to 0$

But $F'''(s) = -\kappa^2\tau \neq 0$

[provided that κ and $\tau \neq 0$]

The osculating plane has 3 point of contact with the curve.

Hence proved

Note:

1. The theory of plane curves it is useful to consider the curvature of a curve and the radius of curvature at 'p'.

2. The radius of circle which has three-point contact with the curve at ‘p’.

3. The radius of curvature is the reciprocal of the curvature in plane curve.

4. The theory of space curves the radius of curvature of γ at ‘p’ defined as the radius of that circle which has three point contact with the curve at ‘p’.

Example 2:

To derive the equation of osculating circle, radius of curvature and the cube of curvature.

[**Osculating circle:**

The osculating circle at a point ‘p’ on a curve at ‘p’. It evidently lies in the osculating plane at ‘p’ and its center $'\bar{c}'$ is at some distance ‘p’ along the principal normal at ‘p’.

$(i.e)\bar{c} - \bar{r} = \rho\bar{n}$

The osculating circle is the section of the sphere $(c - R)^2 - \rho^2 = 0.$]

Proof:

We know that (by definition of osculating circle),

Osculating circle is the circle which has three-point at ‘p’ with the curve.

We know that, osculating plane has three-point at any point of the curve.

∴osculating circle lies on the osculating plane.

(i.e) Osculating circle can be regarded intersects of the osculating plane with the plane curve sphere $(c - R)^2 = a^2$ ………. (1)

Where,

$\bar{R}$ – The general point on the sphere

$\bar{c}$ – The center of the sphere

a – The radius of the sphere

Let $\bar{R} = \bar{r}(s)$ …………. (2) be the equation of the curve T_0.

Get the point of intersection of osculating circle and the curve.

Substituting (2) in (1)

$\therefore [\bar{c} - \bar{r}(s)]^2 = a^2$

Now, the circle will have three-point contact if $F'(s) = F''(s) = 0$ and $F'''(s) \neq 0$.

Where $F(s) = \left(\bar{c} - \bar{r}(s)\right)^2 - a^2$

$F'(s) = 2\left(\bar{c} - \bar{r}(s)\right)\left(-\bar{r}'(s)\right) = 0$

$\left(\bar{c} - \bar{r}(s)\right)\left(\bar{r}'(s)\right) = 0$

(i.e) $(\bar{c} - \bar{r}).\bar{t} = 0$ ………… (3)

⇒(3) implies $\bar{c} - \bar{r}$ lies on the normal plane

But $\bar{c} - \bar{r}$ lies on the osculating plane

$\therefore$ $\bar{c} - \bar{r}$ lies along the normal $\bar{n}$

$\therefore$ $\bar{c} - \bar{r} = \mu\bar{n}$

Substituting this in (4),

$$\mu\bar{n}.\bar{n} = \frac{1}{\kappa}$$

$$\mu = \frac{1}{\kappa} \ \ldots\ldots\ldots..(5)$$

(5) implies $\bar{c} - \bar{r} = \frac{1}{\kappa}\bar{n}$

(i.e) $\bar{c} - \bar{r} = \rho\bar{n} \quad \ldots\ldots\ldots\ldots.(6) \left[\because \rho = \frac{1}{\kappa}\right]$

Equation (6) is the equation of osculating circle $\bar{c}$ in the cube of osculating circle, ρ is the radius of osculating circle and $\bar{c} = \bar{r} + \rho\bar{n}$ gives the cube of curvature.

Remark:

Let $\bar{c} = (\alpha, \beta, \gamma)$

$\bar{n} = (l, m, n)$ and $\bar{n} = (x, y, z)$

$\therefore$ From equation (5) we get,

$$\frac{-x+\alpha}{l} = \frac{(-y+\beta)}{m} = \frac{-z+\gamma}{n} = \rho$$

$\therefore$ $\alpha = x + \rho l, \beta = y + \rho m, \gamma = z + \rho n$ in the coordinates of the cube of curvature at (x, y,z).

Example 3:

Fins the equation of the plane which have three-point contact at the origin with the curve

$\bar{r} = \bar{r}(t^4 - 1, t^3 - 1, t^2 - 1) \quad \ldots\ldots\ldots..(1)$

Proof:

Let the equation of the plane be, $ax + by + cz = 0 \quad \ldots\ldots\ldots\ldots(2)$

Let $F(t) = a(t^4 - 1) + b(t^3 - 1) + c(t^2 - 1)$

Since (1) have three-point contact with (2) at the origin

$\therefore$ $F'(t) = F''(t) = 0$ at $t = 0$ and $F'''(t) \neq 0$ at $t = 0$

$\therefore$ $F'(t) = 4at^3 + 3bt^2 + 2ct$ and

$F''(t) = 12at^2 + 6bt + 2c.$

At the origin $x = 0, y = 0, z = 0,$

$\therefore$ $t = 1,$

$F'(t) = F''(t) = 0.$

$\therefore$ $4a + 3b + 2c = 0$

$12a + 6b + 2c = 0$

$\therefore$ Equation of the required plane is $3x + 8y + 6z = 0$.

Osculating sphere (or) Sphere of curvature:

The osculating sphere which has four-point contact with the curve at p.

If $\bar{c}$ is its center and $\bar{R}$ is radius then the equation of the sphere is $(\bar{c} - \bar{R})^2 = R^2$.

Remark:

$(\bar{c} - \bar{r}).\kappa\bar{n} = 1$ from which $\rho = \kappa^{-1}$

The radius of the osculating circle is $|\rho| = |\kappa^{-1}|$

ρ is called the radius of curvature of the curve at p. Note that $'\rho'$ may be negative.

The center of curvature is the center of the osculating circle and its position vector is given by $\bar{c} = \bar{r} + \rho\bar{n}$.

$\sigma = \tau^{-1}$ is called the radius of Torsion.

Example 4:

Derive the equation of osculating sphere and its center and radius.

Solution:

Let $\bar{c} =$ the center of osculating sphere

And $\bar{R} =$ the position vector of a general point of osculating sphere

$R =$ the radius of osculating sphere.

Then its equation is $(\bar{c} - \bar{R})^2 = R^2 \quad \dots\dots\dots\dots(1)$

Then let the equation of the curve be $\bar{R} = \bar{r}(s) \quad \dots\dots\dots\dots(2)$

To get the point of contact of (1) with (2)

$\therefore F(s) = (\bar{c} - \bar{r})^2 - R^2 = 0 \quad \dots\dots\dots\dots(3)$

Condition for the point to be four-point contact is

$F'(s) = F''(s) = F'''(s) = 0$ and $F^{(iv)}(s) \neq 0$.

Differentiate (3) with respect to 's' we get,

$F'(s) = 2(\bar{c} - \bar{r})(-\bar{r}') = 0$

$(\bar{c} - \bar{r}).\bar{t} = 0 \quad \dots\dots\dots\dots(4)$

$F''(s) = 2[-\bar{t}.\bar{t} + (\bar{c} - \bar{r}).\kappa\bar{n}] = 0$

$[-1 + (\bar{c} - \bar{r}).\kappa\bar{n}](2) = 0$

$[-1 + (\bar{c} - \bar{r})\kappa\bar{n}] = 0$

$(\bar{c} - \bar{r})\kappa\bar{n} = 1$

$(\bar{c} - \bar{r}).\bar{n} = \frac{1}{\kappa} = \rho$

$(\bar{c}-\bar{r}).\bar{n} = \rho$ ... ... (5)

And $F'''(s) = 0 + [(-\bar{t})(\kappa\bar{n}) + (\bar{c}-\bar{r})\kappa'\bar{n} + (\bar{c}-\bar{r})\kappa(\tau\bar{b} - \kappa\bar{t})] = 0$

$[\therefore \bar{n}' = \tau\bar{b} - \kappa\bar{t}]$

$\kappa'(\bar{c}-\bar{r}).\bar{n} + \kappa[\tau(\bar{c}-\bar{r})\bar{b} - \kappa(\bar{c}-\bar{r})\bar{t}] = 0$

$\kappa'(\bar{c}-\bar{r}).\bar{n} + \kappa\tau(\bar{c}-\bar{r}).\bar{b} - 0 = 0$

$[\because (4)]$

(i.e) $\kappa'(\bar{c}-\bar{r}).\bar{n} + \kappa\tau(\bar{c}-\bar{r}).\bar{b} = 0$ ... ... (6)

$$(-\bar{r})\bar{b} = \frac{\kappa'(\bar{c}-\bar{r})\bar{n}}{\kappa\tau}$$

Equation (4) show that $(\bar{c}-\bar{r})$ in perpendicular to $\bar{t}$

It lies on the normal plane.

$\bar{c}-\bar{r}$ can be expressed as

$\bar{c}-\bar{r} = \lambda\bar{b} + \mu\bar{n}$ ... ... (7)

Taking dot product of (7) with $\bar{n}$,

$(\bar{c}-\bar{r}).\bar{n} = (\lambda\bar{b} + \mu\bar{n}).\bar{n}$

$\rho = \mu\bar{n}.\bar{n}$ $[\because (5)]$

$\rho = \mu$

Taking dot product of (7) with $\bar{b}$,

$(\bar{c}-\bar{r}).\bar{b} = (\lambda\bar{b} + \mu\bar{n}).\bar{b}$

$\frac{(\bar{c}-\bar{r}).(-\kappa'.\bar{n})}{\kappa\tau} = \lambda\bar{b}.\bar{b}$ $[\because \bar{n}.\bar{b} = 0]$

$\frac{-\kappa'(\bar{c}-\bar{r})\bar{n}}{\kappa\tau} = \lambda$ $[\therefore (6)]$

$\frac{-\kappa'}{\kappa\tau}\rho = \lambda$ $[\because (5)]$

Substituting λ and μ in (7) we get,

$\bar{c}-\bar{r} = \frac{-\kappa'\rho}{\kappa\tau}\bar{b} + \rho\bar{n}$ $[\because \rho = \frac{1}{\kappa}]$

$= \rho\bar{n} - \frac{\kappa'}{\kappa\tau}\rho\bar{b}$

$\bar{c}-\bar{r} = \rho^{\bar{n}} + \left(\frac{-\kappa'}{\kappa^2}\right)\frac{1}{\tau}\bar{b}$

$= \rho\bar{n} + \sigma p'\bar{b}$

(ie) $\bar{c} = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b}$

This gives the center of osculating plane (sphere) It is called as center of Spherical Curvature

$$
\begin{aligned}
(\bar{c}-\bar{r})^2 &= (\bar{c}-\bar{r})\cdot(\bar{c}-\bar{r})\\
&= \left(\rho\bar{n}+\sigma\rho'\bar{b}\right)\cdot\left(\rho\bar{n}+\sigma\rho'\bar{b}\right)\\
&= \rho^2+(\sigma\rho')^2\ [\because (\bar{c}-\bar{r})^2 = R^2]
\end{aligned}
$$

Radius of the osculating sphere in

It is radius of Spherical curvature.

If κ is constant, then $\rho' = 0$

$\Rightarrow\ \bar{c}-\bar{r} = \rho[\because \bar{c}-\bar{r} = R]$

Note:

1. Q: Prove that the osculating sphere at a point on a curve & derive the formula for its center and radius (or)

   Define osculating sphere and obtain expression for center of curvature and radius of curvature.

   (or)

   Prove that the center of curvature of the pt. $\bar{\gamma}$ of gen. curve in the pt. $\bar{r}+\rho\bar{n}+\rho'\bar{r}\bar{b}$.
2. Center of spherical curvature $= \bar{c}-\bar{r} = \rho\bar{n}+\sigma p'\bar{b}$
3. position vector $\bar{c} = \bar{r}+p\bar{n}+\sigma\rho'\bar{b}$
4. Radius of Spherical curvature $= R = (\rho^2+\sigma^2 p'^2)^{\frac{1}{2}}$

Locus of the center of Spherical curvature:

As the point p traces out a curve c, the corresponding center of spherical curvature traces out another curve c_1, whose curvature and torsion are simply related to the curvature and torsion of the original curve c.

Example 1:

To obtain the focus of center of spherical curvature. (OR) Relation of c & C bet'n the curvature & Torsion.

Proof:

Let c be the given curve and c_1 be the locus of center of the osculating sphere.

Let the suffice unity denote the corresponding quantities for the locus c.

Thus $\bar{r}_1$ denote the position vector of a general point on c_1.

(i.e) $\bar{r}_1$, is the position vector of center of spherical curvature

So, $\bar{r}_1 = \bar{r}+\rho\bar{n}+\rho'\sigma\bar{b}$ ……………(1)$\left[\because \rho\kappa = 1 \Rightarrow \rho = \frac{1}{\kappa}, \tau = \frac{1}{\sigma}\right]$

Differentiate (1) with respect to 's'

$\frac{d\bar{r}_1}{ds} = \bar{r}' + \rho\bar{n}' + \rho'\bar{n} + \sigma'\rho'\bar{b} + \rho''\sigma\bar{b} + \rho'\sigma\bar{b}'$

$\frac{d\bar{r}_1}{ds_1} \cdot \frac{ds_1}{ds} = \bar{t} + \rho(\tau\bar{b} - \kappa\bar{t}) + \rho'\bar{n} + (\rho'\sigma)'\bar{b} + \rho'\sigma(-\tau\bar{n})$

$\frac{ds_1}{ds}\bar{t}_1 = \bar{t} + \rho\tau\bar{b} - \rho\kappa\bar{t} + \rho'\bar{n} + (\rho'\sigma)'\bar{b} - \rho'\sigma\tau\bar{n}$

$= \bar{t} + \rho\tau\bar{b} - (1)\bar{t} + \rho'\bar{b} + (\rho'\sigma)'\bar{b} - \rho'\sigma\frac{1}{\sigma}\bar{n}$

$= \bar{t} + \rho\tau\bar{b} - \bar{t} + \rho'\bar{n} + (\rho'\sigma)'\bar{b} - \rho'\bar{n}$

$= \rho\tau\bar{b} + \rho''\sigma\bar{b} + \rho'\sigma'\bar{b}$

$s_1'\bar{t}_1 = [\rho\tau + \rho''\sigma + \rho'\sigma']\,\bar{b}$

Squaring (2) on both sides,

$(s_1', \bar{t}_1)^2 = \left[[\rho\tau + \rho''\sigma + \rho'\sigma']\bar{b}\right]^2$

$\Rightarrow \quad (s_1')^2 = [\rho\tau + \rho''\sigma + \rho'\sigma'] \ldots\ldots\ldots..(3)$

Equation (2) shows that $\bar{t}_1$ is parallel to $\bar{b}$.

∴ Let $\overline{t_1} = \rho\bar{b}$--------- (4) where $\rho = \pm 1$.

Differentiate (4) with respect to ' s '.

$\frac{d\bar{t}_1}{ds} = \rho\bar{b}$

$\Rightarrow \frac{d\bar{t}_1}{ds_1} \cdot \frac{ds_1}{ds} = \rho(-\tau\bar{n})$

$\Rightarrow (\kappa_1\bar{n}_1) \cdot s_1' = -\rho\tau\bar{n} \quad \ldots\ldots\ldots\ldots (5)$

Equation (5) shows that $\bar{n}_1$ is parallel to $\bar{n}$

∴ Let $\bar{n}_1 = \rho_1\bar{n} \ldots\ldots\ldots..(6)$ where $\rho_1 = \pm 1$

Substituting (6) in (5),

$(5) \Rightarrow \kappa_1\rho_1 s_1'\bar{n} = -\rho\tau\bar{n}.$

$\Rightarrow \kappa_1\rho_1 s_1' = -\rho\tau$

$\Rightarrow \; s_1' = \frac{-\rho\tau}{\kappa_1\rho_1} \ldots\ldots\ldots\ldots (7)$

We know that, $\bar{b}_1 = \bar{t}_1 \times \bar{n}_1$ [$\because \bar{b} = \bar{t} \times \bar{n}$]

$= \rho\bar{b} \times \rho_1\bar{n}$ [by (4) & (6)]

$= \rho\rho_1(\bar{b} \times \bar{n})$ [$\because \bar{n} \times \bar{b} = \bar{E}$

$(\text{i.e})\bar{b}_1 = -\rho\rho_1\bar{t} \quad \ldots\ldots\ldots\ldots.(8)[\because \bar{b} \times \bar{n} = -\bar{t}]$

∴ $\bar{b}_1$ is parallel to $\bar{t}$

∴ Differentiate (8) with respect to 's',

$\frac{d}{ds}(\bar{b}_1) = -\rho\rho_1\bar{t}'$

$\frac{d}{ds}(\bar{b}_1)\frac{ds_1}{ds} = -\rho\rho_1(\kappa\bar{n})$

$\bar{b}_1' s_1' = -\rho\rho_1(\kappa\bar{n})$

$-\tau_1\bar{n}_1 s_1' = -\rho\rho_1\kappa\bar{n}$

$\tau_1\rho_1\bar{n}s_1' = \rho\rho_1\kappa\bar{n}$

$\Rightarrow\ \tau_1 s_1' = \rho\kappa$

$\Rightarrow\ s_1' = \frac{\rho\kappa}{\tau_1}$

From (7) & (9) we get,

$-\frac{\rho\tau}{k_1\rho_1} = \frac{\rho\kappa}{\tau_1}$

$\Rightarrow \frac{-\tau}{k_1\rho_1} = \frac{\kappa}{\tau_1}$

$\Rightarrow \tau\tau_1 = -kk_1\rho_1$

if $\rho_1 = -1$,

(i.e) The product of the torsions } = $\left\{\begin{array}{l}\text{The product of} \\ \text{the curvature it } \rho_1 = -1 \text{ at corresponding points}\end{array}\right.$

then $\frac{\kappa}{\tau_1} = \frac{\tau}{\kappa_1}$

(ie) $\kappa\kappa_1 = \tau\tau_1$ which is required result

when $\rho_1 = -1$ & $\rho = -1$

$$\frac{ds_1}{ds} = s_1' = \frac{\kappa}{\tau_1} = \frac{\tau}{\kappa_1} \ \ldots\ldots\ldots\ldots..(*)$$

Note:

If $\rho\sigma^{-1} + \sigma'\rho' + \sigma\rho'' = 0$, then then corresponding point on ρ_1 in a singular point.

Theorem 1:

If the curvature κ of c is constant, then the curvature κ_1 of c_i is also constant.

Proof:

Since κ is constant.

$\Rightarrow \rho = \frac{1}{\kappa}$ is also constant.

$\therefore\ \rho' = \rho'' = 0$

$\therefore$ From equation (3) we have,

$$\left(\frac{ds_1}{ds}\right)^2 = [\rho\tau + (\rho'\sigma)']^2$$
$$= [\rho\tau + 0]^2$$
$$= \rho^2\tau^2$$
$$\frac{ds_1}{ds} = \rho\tau$$
$$= \frac{\tau}{k}\left[\because \rho = \frac{1}{\kappa}\right]$$
$$\Rightarrow \frac{ds_1}{ds} = \frac{\tau}{k}$$

(But from (*))

Thus $\frac{\tau}{\kappa} = \frac{\tau}{\kappa_1} \Rightarrow \kappa = \kappa_1$

(i.e) κ_1 is also constant.

Theorem 2:

The radius of curvature ρ_1 & radius of torsion σ_1 are given by.

$\rho_1 = \rho + \sigma\frac{d}{ds}(\sigma p')$ and $\sigma_1 = \frac{\rho^2}{\sigma} + \rho\frac{d}{ds}(\sigma\rho')$

Proof:

From equation (*)

$$\frac{ds_1}{ds} = \frac{\tau}{\kappa_1}$$
$$\therefore \frac{1}{\kappa_1} = \frac{1}{\tau}\cdot\frac{ds_1}{ds} \quad \text{(using eqn (3))}$$
$$= \sigma\left[\frac{\rho}{\sigma} + (\sigma\rho')'\right]$$
$$\rho_1 = \rho + \sigma\frac{d}{ds}(\sigma\rho')$$

Similarly using (*),

$$\sigma_1 = \frac{\rho^2}{\sigma} + \rho\frac{d}{ds}(\sigma\rho')$$

Example 2:

Prove that the radius of curvature of the locus of the center of curvature of a curve is given by,

$$\left[\left\{\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{R}\right\}^2 + \frac{\rho'\sigma_4}{\rho^2R^4}\right]^{-1/2}$$

where ρ, σ, R have the usual meaning.

Proof:

Let the suffix unity be used to distinguish quantities belonging to the locus of the center of curvature of the given curve.

$\therefore$ If $\bar{r}_1$ denotes the position vector, the center of curvature then,

$\bar{r}_1 = \bar{r} + \rho\bar{n}$ .... (1)

Differentiate with respect to 's', we get,

$\frac{d\bar{r}_1}{ds_1} \cdot \frac{ds_1}{ds} = \bar{r}' + \rho'\bar{n} + \rho\bar{n}'$

$\frac{ds_1}{ds} E_1 = \bar{t} + \rho'\bar{n} + \rho(\tau\bar{b} - \kappa\bar{t})$

$= \bar{t} + \rho'\bar{n} + \rho\tau\bar{b} - \rho\kappa\bar{t} \left[\because \rho = \frac{1}{\kappa}\right]$

$= \bar{t} + \rho'\bar{n} + \frac{\rho}{\sigma}\bar{b} - \bar{t} \left[\tau = \frac{1}{\sigma}\right]$

$= \rho'\bar{n} + \frac{\rho}{\sigma}\bar{b}$

$\Rightarrow \frac{ds_1}{ds}\bar{t}_1 = \frac{\rho}{\sigma}\left[\frac{\sigma}{\rho}\rho'\bar{n} + \bar{b}\right]$

$\Rightarrow \frac{\sigma}{\rho} \cdot \frac{ds_1}{ds}\bar{t}_1 = \frac{\sigma}{\rho}\rho'\bar{n} + \bar{b}$ ... ... (2)

Squaring (2) we get,

$\Rightarrow \left[\frac{\sigma}{\rho}\frac{ds_1}{ds}\bar{t}_1\right]^2 = \left[\frac{\sigma}{\rho}\rho'\bar{n} + b\right]^2$

$\Rightarrow \frac{\sigma^2}{\rho^2}\left(\frac{ds_1}{ds}\right)^2 (1) = \frac{\sigma^2}{\rho^2}\rho'^2(1) + (1) + 2\frac{\sigma\rho'}{\rho} \cdot \tilde{n} \cdot \bar{\text{b}}$

$= \frac{\sigma^2}{\rho^2}\rho^2 + 1 \; [\because \bar{n}^2 = \bar{n} \cdot \bar{n} = 1 \; and \; \bar{n} \cdot \bar{b} = 0, \bar{t}^2 = 1]$

(i.e) $\frac{\sigma^2}{\rho^2}\left(\frac{ds_1}{ds}\right)^2 = \frac{\sigma^2\rho'^2 + \rho^2}{\rho^2}$

$\Rightarrow \left(\frac{ds_1}{ds}\right)^2 = \frac{1}{\sigma^2}(\sigma^2\rho'^2 + \rho^2) = \frac{R^2}{\sigma^2} [\because R^2 = \sigma^2\rho'^2 + \rho^2]$

Where $R =$ The radius of the osculating sphere.

$\therefore \frac{ds_1}{ds} = \frac{R}{\sigma}$ ... ... (3)

Differentiate (2) with respect to 's' provides,

$L \cdot H \cdot s = \frac{\sigma}{\rho} \cdot \frac{ds_1}{ds} \cdot \frac{dt_1}{ds} + \frac{d}{ds}\left[\frac{\sigma}{\rho} \cdot \frac{ds_1}{ds}\right]\bar{t}_1$

$= \frac{\sigma}{\rho} \cdot \frac{ds_1}{ds} \cdot \frac{dt_1}{ds_1} \cdot \frac{ds_1}{ds} + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{ds_1}{ds}\right]\bar{t}_1$

$= \frac{\sigma}{\rho}\left[\frac{ds_1}{ds}\right]^2 \frac{dt_1}{ds_1} + \frac{d}{ds}\left[\frac{\sigma}{\rho} \cdot \frac{ds_1}{ds}\right]\bar{t}_1$

$$= \frac{\sigma}{\rho}\left[\frac{ds_1}{ds}\right]^2 \kappa_1 \bar{n}_1 + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{ds_1}{ds}\right]\bar{t}_1 \dots\dots\dots\dots (4)$$

$$R \cdot H \cdot S = \frac{\sigma}{\rho}\rho' \frac{d\bar{n}}{ds} + \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right]\bar{n} + \bar{b}'$$

$$= \frac{\sigma}{\rho}\rho'[\tau\bar{b} - \kappa t] + \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right]\bar{n} - \tau\bar{n}$$

$$= \frac{\rho'\sigma}{\rho}\tau\bar{b} - \frac{\sigma\rho'}{\rho}\kappa\bar{t} + \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right]\bar{n} - \tau\bar{n}$$

$$= \frac{\rho'\sigma}{\rho}\tau b - \frac{\sigma\rho'}{\rho}\kappa t + \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \tau\right]\bar{n} \quad \dots\dots\dots\dots\dots (5)$$

$$\therefore \frac{\sigma}{\rho}\left[\frac{ds_1}{ds}\right]^2 \kappa_1 \bar{n}_1 + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{ds_1}{ds}\right]\bar{t}_1 = -\frac{\sigma\rho'}{\rho}\kappa t + \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \tau\right]\bar{n} + \frac{\rho'\sigma}{\rho}\tau b \quad \dots\dots\dots\dots (6)$$

∴ The vector (×) product of (2) and (6),

$$\frac{\sigma}{\rho}\frac{ds_1}{ds}\bar{t}_1 \times \left[\frac{\sigma}{\rho}\left[\frac{ds_1}{ds}\right]^2 \kappa_1 \bar{n}_1 + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{ds_1}{ds}\right]\bar{t}_1\right]$$

$$= \left[\frac{\sigma}{\rho}\rho'\bar{n} + \bar{b}\right] \times \left[-\frac{\sigma\rho'}{\rho}\kappa t + \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \tau\right]\bar{n} + \frac{\rho'\sigma}{\rho}\tau b\right]$$

$$\frac{\sigma^2}{\rho^2}\left(\frac{ds_1}{ds}\right)^3 \kappa_1(\bar{b}) + 0 = \frac{-\sigma^2}{\rho^2}\rho'^2\kappa(-\bar{b}) + \sigma + \frac{\sigma^2}{\rho^2}\rho'^2\tau\bar{t} - \frac{\sigma\rho'}{\rho}\kappa(\bar{n}) - \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \tau\right]\bar{t} + 0$$

$$= +\frac{\sigma^2}{\rho^2}\rho'^2\kappa\bar{b} + \frac{\sigma^2}{\rho^2}\rho'^2\tau\bar{t} - \frac{\sigma\rho'}{\rho}\kappa\bar{n} - \left[\frac{ds}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \tau\right]\bar{t}$$

$$\frac{\sigma^2}{\rho^2}\left(\frac{ds_1}{ds}\right)^3 \kappa_1\bar{b} = \frac{\sigma^2}{\rho^2}\rho'^2\kappa\bar{b} - \frac{\sigma\rho'}{\rho}\kappa\bar{n} + \left[\frac{\sigma^2\rho'^2}{\rho^2}\tau - \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] + \tau\right]\bar{t}$$

Squaring on both sides,

$$\frac{\sigma_4}{\rho^4}\left(\frac{ds_1}{ds}\right)^6 \kappa_1^2 = \frac{\sigma 4}{\rho^4}\rho'^4\kappa^2 + \sigma^2\frac{\rho'^2}{\rho^2}\kappa^2 + \left[\frac{\sigma^2\rho'^2}{\rho^2}\tau - \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] + \tau\right]$$

$$\frac{R^6}{\sigma^2\rho^4\rho_1^2} = \frac{\sigma 4}{\rho^4}\rho^4\kappa^2 + \frac{\sigma^2\rho'^2}{\rho^2}\kappa^2 + \left[\frac{\sigma^2\rho'^2}{\rho^2} - \frac{1}{\sigma} - \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] + \frac{1}{\sigma}\right]^2$$

$$= \frac{\sigma^4}{\rho^4}\rho'^4 \cdot \kappa^2 + \frac{\sigma^2\rho'^2}{\rho^2}\kappa^2 + \left[\frac{\sigma^2\rho^2 + \rho^2}{\sigma\rho^2} - \frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right]\right]^2$$

$$= \frac{\rho^2\sigma^4\rho'^4\kappa^2 + \rho^4\sigma^2\rho'^2\kappa^2}{\rho^6} + \left[\frac{d}{ds}\left[\frac{\sigma\rho^1}{\rho}\right] - \left[\frac{\sigma^2\rho'^2 + \rho^2}{\rho^2}\right]\right]$$

$$= \frac{\rho^2\sigma^2\rho'^2\kappa^2[\sigma^2\rho'^2 + \rho^2]}{\rho^4} + \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \left[\frac{\sigma^2\rho'^2 + \rho^2}{\sigma\rho^2}\right]\right]$$

$$\frac{R^6}{\sigma^2\rho^4\rho_1^2} = \frac{\sigma^2\rho'^2[\sigma^2\rho'^2 + \rho^2]}{\rho^6} + \left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right] - \left[\frac{\sigma^2\rho^2 + \rho^2}{\sigma\rho^2}\right]\right]^2$$

$$\frac{1}{\rho_1^2} = \frac{\rho^4\sigma^2}{R^6}\left[\frac{\sigma^2 p^2[\sigma^2 p^2+\rho^2]}{\rho^2}\right] + \frac{\rho^4\sigma^2}{R^6}\left[\frac{d}{ds}\left[\frac{\sigma p'}{\rho}\right] - \frac{R^2}{\sigma p^2}\right]^2 \ [\because R^2 = \sigma^2\rho'^2 + \rho^2]$$

$$= \frac{\sigma_4\rho'^2}{\rho^2 R^6}[R^2] + \frac{\rho^4\sigma^2}{R^6}\left[\frac{d}{ds}\left[\frac{\sigma p'}{\rho}\right] - \frac{R^2}{\sigma p^2}\right]^2$$

$$\frac{1}{\rho_1^2} = \frac{\sigma^4\rho^2}{\rho^2 R^4} + \left[\frac{\rho^2\sigma}{R^3}\left[\frac{d}{ds}\left[\frac{\sigma\rho}{\rho^2}\right]\right] - \frac{\rho^2\sigma}{R^3}\cdot\frac{R^2}{\sigma p^2}\right]^2$$

$$\frac{1}{\rho_1^2} = \frac{\sigma^4 p'^2}{\rho^2 R^4} + \left[\frac{\rho^2\sigma}{R^3}\left[\frac{d}{ds}\left[\frac{\sigma\rho'}{\rho}\right]\right] - \frac{1}{R}\right]^2$$

$$\therefore \rho_1 = \frac{\sigma^4 C'^2}{R^2 R^4} + \left[\frac{\rho^2\sigma}{R^3}\left[\frac{d}{ds}\left[\frac{\sigma\rho'}{P}\right]\right] - \frac{1}{R}\right]^{-1/2}$$

Example 3:

If the radius of spherical curvature is constant, prove that the curve either lies on a sphere or has constant curvature.

Proof:

We know that Radius of the spherical curvature is given by,

$R^2 = \rho^2 + (\rho'\sigma)^2$ .... (1)

Given R is constant $\Rightarrow \frac{dR}{ds} = 0$.

Differentiate (1) with respect to 's',

$$2R\cdot\frac{dR}{ds} = 2\rho\rho' + 2(\rho'\sigma)(\rho'\sigma)'$$

$$0 = 2\rho'\left[\rho + \sigma\frac{d}{ds}(\rho'\sigma)\right]$$

$\Rightarrow$ either $2\rho' = 0$ (or) $\rho^+\sigma\frac{d}{ds}(\rho'\sigma) = 0$.

If $\rho' = 0 \Rightarrow \rho =$ constant.

$\Rightarrow \frac{1}{k} =$ constant

(i.e) the curve has a constant curvature.

If $\rho + \sigma\frac{d}{ds}[\rho'\sigma] = 0$.

then $\sigma\left[\frac{\rho}{\sigma} + \frac{d}{ds}[\rho'\sigma]\right] = 0$.

$\Rightarrow \frac{p}{\sigma} + \frac{d}{ds}[\rho'\sigma] = 0$

$\Rightarrow$ A curve lies on a Sphere [by $Ex(3)$]

Example 4:

If a curve lies on a sphere. Show that ρ & σ ane related by $\frac{d}{ds}[\sigma, \rho'] + \frac{\rho}{\sigma} = 0.$ (or)

Show that N and s condition that a curve lies on a sphere is that $\frac{\rho}{\sigma} + \frac{d}{ds}\left[\frac{\rho'}{\tau}\right] = 0$ at every point of the curve.

Proof:

Necessary part:

Assume that the curve lies on a sphere.

∴ The osculating sphere of every point of the curve is nothing but the given sphere itself

∴ R is the radius of osculating sphere (i.e) The radius of sphere curve is constant

We know that, $R^2 = \rho^2 + \rho'^2\sigma^2$ ……………(1)

$[R = constant\ implies \frac{dR}{ds} = 0]$

Differentiate (1) with respect to 's',

$$2R\frac{dR}{ds} = 2\rho\rho' + 2\rho'\rho''\sigma^2 + 2\rho'^2\sigma\sigma'$$
$$= 2\rho'[\rho'\rho''\sigma^2 + \rho'\sigma'\sigma']$$
$$0 = 2\rho'[\rho + \sigma[\rho''\sigma + \rho'\sigma']]$$
$$0 = 2\rho'\left[\rho + \sigma\frac{d}{ds}(\rho'\sigma)\right]$$
$$0 = 2\rho'\sigma\left[\frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma)\right]$$
$$\Rightarrow \frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma) = 0\ [\because \sigma \neq 0]$$

This is required condition.

Sufficient part:

Let $\frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma) = 0$ … … … … … (2)

$\Rightarrow \sigma\left[\frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma)\right] = 0\ [\because \sigma \neq 0]$

(ie) $\rho + \sigma\frac{d}{ds}(\rho'\sigma) = 0.$

(ie) $2\rho'\left[\rho'\sigma'\frac{d}{ds}(\rho'\sigma)\right] = 0.$

(or) $\rho\rho' + (\rho'\sigma)\frac{d}{ds}(\rho'\sigma) = 0.$

$\Rightarrow \frac{d}{ds}[\rho^2 + (\rho'\sigma)^2] = 0.$

$\Rightarrow \rho^2 + (\rho'\sigma)^2 =$ constant

$\Rightarrow R^2 =$ constant.

Where $R =$ radius of the osculating sphere. Thus the radius of the osculating sphere is constant

We know that,

The cube of osculating sphere is given by,

$\bar{c} = \bar{r} + \rho\bar{n} + \rho'\sigma\bar{b}$

Differentiate with respect to 's',

$\frac{d\bar{c}}{ds} = \bar{r}' + \rho'\bar{n} + \rho\bar{n}' + (\rho'\sigma)'\bar{b} + \rho'\sigma\bar{b}'.$

$\frac{d\bar{c}}{ds} = \bar{t} + \rho'\bar{n} + \rho(\tau\bar{b} - \kappa\bar{t}) + (\rho'\sigma)'\bar{b} + (\rho'\sigma)(-\tau\bar{n})$

$= \bar{t} + \rho'\bar{n} + \rho\tau\bar{b} - \rho\kappa\bar{t} + (\rho'\sigma)'\bar{b} + (\rho'\sigma)(-\tau\bar{n})$

$= \bar{t} + +\rho'\bar{n} + \rho\tau\bar{b} - \rho\left(\frac{1}{\kappa}\right)\bar{t} + (\rho'\sigma)'\bar{b} + (\rho'\sigma)(-\tau\bar{n})$

$= \rho'\bar{n} + \rho\tau\bar{b} + (\rho'\sigma)'\bar{b} - \rho'\sigma\cdot\frac{1}{\sigma}\bar{n}$

$= \rho'\tau\bar{b} + (\rho'\sigma)'\bar{b}$

$\frac{d\bar{c}}{ds} = \left[\rho\tau + \frac{d}{ds}(\rho'\sigma)\right]\bar{b}$ [by (2)].

$\Rightarrow \frac{d\bar{c}}{ds} = 0 \Rightarrow \bar{c} = constant$

(i.e) cube of the osculating sphere in constant

From (3)and (4)

We conclude that, The osculating sphere is same at every point of the curve.

∴ The curve lies on a sphere.

1.7.Tangent Surface, Involutes And Evolutes:

A tangent space curve C determines two infinite systems of curves which are the invoutes and evolutes of 'C'

The theory of evolutes of space curves is essentially different from that or plane curve. a plane curve has a unique evolute while a space curve has infinitely many, the evolute of a plane curve is offer defined as the locus of its center of curvature but it will be seen that neither the locus of the center of curvature nor the locus of the center of Spherical curvature are evolutes of a space curve.

A natural generalization to space curves of the concept of involute of a plane curve and once an involute of a curve $\bar{c}$ has been defined, it is natural to define C to be an evolute of $\bar{c}$.

Tangent Surface:

The tangent surface of a curve C is the surface generated by lines tangent to c. Any point. 'p' on the tangent surface is determined by two parameters ' s ' and ' u '.

Where ' s ' is the arc length of C measured from some convenient base point on the curve to a point where the tangent pass through P and ' u ' measures the distance of ' p ' along this tangent.

The position vector of ' p ' can be written as

$\bar{R}(s,u) = \bar{r}(s) + u\bar{t}(s)$

Additional relation bet ' n ' u ' and ' s ' of the form $u = \lambda(s)$ determines a curve which lies on the tangent surface of ' c '.

The class of the curve being the same as the class of λ (or) C, whichever in the Smaller.

Involute:

An involute of c is a curve ' c_1 ' which lies on the tangent surface of c and intersects the generators orthogonally.

Example 1:

To derive the equation of involute of the given curve.

Solution:

Let ' c ' be a given curve with eqn. $\bar{r} = \bar{r}(s)$. Let $'c_1'$ be the involute of ' c '.

we shall use the suffix unity to denote the quantities belonging to c_1

Let P_1 be an arbitrary $point$ on c_1 then $\overline{OP_1} = \overline{OP} + \overline{PP_1}$ $[\bar{R} = r_1]$.

$r_1 = \bar{r} + \lambda(s)\bar{t}$

Differentiate (1) with respect to ' s_1'

$\frac{dr_1}{ds_1} = \frac{d}{ds_1}[r_1 + \lambda(s)t]$

$= \frac{d}{ds}[\bar{r} + \lambda(s)\bar{t}]\frac{ds}{ds_1}$

(i.e) $\quad \bar{t}_1 = \left[\frac{d\bar{r}}{ds} + \lambda'\bar{t} + \lambda\bar{t}'\right]\frac{ds}{ds_1}$

$I_1 = [\bar{t} + \lambda'\bar{t} + \lambda\bar{t}']\frac{ds}{ds_1}$ ... ... (2)

Taking dot product of (2) with $\bar{t}$,

$\bar{t}_1 \cdot \bar{t} = (1 + \lambda')\frac{ds}{ds_1}$

But $\bar{t}_1$ is perpendicular to $\bar{t}$,

$\therefore 0 = (1 + \lambda')\frac{ds}{ds_1}$

$\Rightarrow \frac{ds}{ds_1} \neq 0 \therefore (1 + \lambda') = 0 \Rightarrow \lambda' = -1$

$\frac{d\lambda}{ds} = -1$

(or) $d\lambda = -ds$

$\Rightarrow \lambda = -s + c$

where c = constant.

Substituting in (1)

$\therefore (1) \Rightarrow \bar{r}_1 = \bar{r} + (c - s)\bar{t}$

(i.e) $\bar{R} = \bar{r} + (c-5)\bar{t}$

This is equation of involute of ' c '

This equation represents an infinite system of invotutes of ' c ', a different curve arising from each different choice of the parameter ' c '.

Example 2:

To derive an expression the curvature & torsion of the involute, (or) Show that the torsion of an involute of a curve $= \frac{\rho(\sigma\rho^1 - \sigma'\rho)}{(\rho^2+\sigma^2)(c-s)}$

Solution:

We know that the equation of the involute

$\overline{r_1} = \bar{r} + (c-s)\bar{t}$ ……………(1)

Differentiate (1) with respect to ' s_1 ',

$$\frac{d}{ds_1}(\bar{r}_1) = \frac{d}{ds_1}(\bar{r} + (c-s)\bar{t}] = \frac{d}{ds}[\bar{r} + (c-s)\bar{t}]\frac{ds}{ds_1}$$

$$t_1 = \left[\frac{d\bar{r}}{ds} + (-1)\bar{t} + (c-s)\bar{t}'\right]\frac{ds}{ds_1}\left(\because \frac{d\bar{r}}{ds} = \bar{r}' = \bar{t}\right)$$

$$t_1 = [(c-s)\kappa\bar{n}]\left(\frac{ds}{ds_1}\right) \text{………… } (2)$$

Taking dot product of (2) with itself.

$$\bar{t}_1 \cdot \bar{t}_1 = \left[c(-s)\kappa\bar{n}\frac{ds}{ds_1}\right]\cdot\left[(c-s)\kappa\bar{n}\frac{ds}{ds_1}\right].$$

$$1 = (c-s)^2\kappa^2\left(\frac{ds}{ds_1}\right)^2$$

$$(\frac{ds}{ds_1}0^2 = \frac{1}{(c-s)^2\kappa^2} \text{ ………… } (3)$$

(2) shows that $\bar{t}_1$ is parallel to $\bar{n}$.

Differentiate (2) with respect to s_1'

$$\frac{dt_1}{ds_1} = \frac{d}{ds_1}\left[((c-s)\kappa\bar{n})\left(\frac{ds}{ds}\right)\right]$$

$$\bar{t}'_1 = \frac{d}{ds}[(c-s)\kappa\bar{n})\Big]\frac{ds}{ds_1}$$

$$\kappa_1\bar{n}_1 = \left[(-1)\kappa\bar{n}\left(\frac{ds}{ds_1}\right) + (c-s)\kappa\bar{n}'\left(\frac{ds}{ds_1}\right)\right]\frac{ds}{ds_1}$$

$$\kappa_1\bar{n}_1 = [-\kappa\bar{n} + (c-s)\kappa(\tau\bar{b} - \kappa\bar{t})]\left(\frac{ds}{ds_1}\right)^2$$

$$\kappa_1\bar{n}_1 = (c-s)\kappa(\tau\bar{b} - \kappa\bar{t})\cdot\frac{1}{(c-s)^2\kappa^2}$$

$$\kappa_1\bar{n}_1 = \frac{\tau\bar{b} - \kappa\bar{t}}{(c-s)\kappa} \text{ ……………. } (4)$$

Taking dot product of (4) with itself,

$(\kappa_1 \bar{n}_1) \cdot (\kappa_1 . \bar{n}_1) = \left[\frac{\tau\bar{b}-\kappa\bar{t}}{(c-s)\kappa}\right] \cdot \left[\frac{\tau\bar{b}-\kappa\bar{t}}{(c-s)\kappa}\right]$

$\kappa_1^2 = \frac{\tau^2+\kappa^2}{(c-s)^2 k^2}$ ... ... (5)

We know that, $\overline{b_1} = \bar{t}_1 \times \overline{n_1}$

$\overline{b_1} = \bar{n} \times \left[\frac{\tau\bar{b}-\kappa\bar{t}}{\kappa\kappa_1(c-s)}\right]$ [$\because t_1$ is parallel to $\bar{n}$]

(i.e) $\kappa\kappa_1(c-s)\bar{b}_1 = \tau(\bar{n} \times \bar{b}) - \kappa(\bar{n} \times \bar{t})$

$\kappa\kappa_1(c-s)\bar{b}_1 = \tau\bar{t} + k\bar{b}$ ... ... (6)

Differentiate (6) with respect to ' s '

$\frac{d}{ds_1}\left[\kappa\kappa_1(c-s)\bar{b}_1\right] = \frac{d}{ds_1}\left[\tau\bar{t} + \kappa\bar{b}\right]$

$\left\{\frac{d}{ds_1}\left[\kappa\kappa_1(c-s)\right]\right\}\bar{b}_1 + (\kappa\kappa_1)(c-s)\bar{b}_1' = \frac{d}{ds}\left[\tau\bar{f} + k\bar{b}\right]\left(\frac{ds}{ds_1}\right)$

$\left\{\frac{d}{ds_1}\left[\kappa\kappa_1(c-s)\right]\right\}\bar{b}_1 + (\kappa\kappa_1)(c-s)\tau_1\bar{n}_1 = \left[\tau'^{\bar{t}} + \tau\bar{t}' + \kappa'\bar{b} + \kappa\bar{b}'\right]\left(\frac{ds}{ds_1}\right)$

$\left\{\frac{d}{ds_1}\left[kk_1(c-s)\right]\right\}\bar{b}_1 - \kappa\kappa_1(c-s)\tau_1\bar{n}_1 = \left[\tau'\bar{t} + \tau k\bar{n} + \kappa'\bar{b} - k\tau\bar{n}\right]\frac{ds}{ds_1}$

$\left\{\frac{d}{ds_1}\left[kk_1(c-s)\right\}\bar{b}_1 - \kappa\kappa_1(c-s)\tau_1\bar{n}_1 = \left[\tau'\bar{t} + \kappa'\bar{b}\right]\frac{ds}{ds_1}\right.$ ... ... (7)

Taking dot product (7) with (4),

$\therefore [\kappa_1 n_1] \cdot \left[\left[\frac{d}{ds_1}(\kappa\kappa_1(c-s))\right]\overline{b_1} - \kappa_1\kappa(c-s)\tau_1\bar{n}_1 = \left[\frac{\tau\bar{b}-\kappa\bar{t}}{(c-s)\kappa}\right] \cdot \left[\frac{\tau'\bar{t}+\kappa'\bar{b}}{(c-s)\kappa}\right]\right.$

$-\kappa\kappa_1^2(c-s)\tau_1 = \frac{1}{\kappa^2(c-s)^2}\left[\tau k' - k\tau'\right]$

$\tau_1 = \frac{\kappa\tau'-\tau\kappa'}{\kappa^3\kappa_1^2(c-s)^3}$

$= \frac{(\kappa\tau'-\tau\kappa')}{\kappa^3(c-s)^3} \cdot \frac{\kappa^2(c-s)^2}{(\tau^2+\kappa^2)}$

$\tau_1 = \frac{(\kappa\tau'-\tau\kappa')}{\kappa(\tau^2+\kappa^2)(c-s)}$ ... ... (8) [$\because$ (5)]

Equation (5) and (8) gives the values of κ_1 & τ_1 of the involute respectively

Evolute:

If $\tilde{c}_1$ in an involute of c, then c is an evolute of $\tilde{c}$.

Example 3:

To derive the equation of evolute.

Solution:

Let $\tilde{c}$ be the curve & c be the evolute of $\tilde{c}$.

Let P to the point on c corresponding to the point Q on $\tilde{c}$.

Then P must lie in the plane through Q normal to $\tilde{c}$.

If $\bar{R}, \bar{r}$ denote the position vectors of P, Q respectively then the equation of $\hat{c}$ is

$$\overline{OQ} = \overline{OP} + \overline{PQ}$$
$$\bar{R} = \bar{r} + \lambda\bar{b} + \mu\bar{n} \quad \ldots\ldots\ldots\ldots..(*)$$

[$\because$ PQ is perpendicular to the tangent at P. So it lies on the normal plane].

Differentiate $(*)$ with respect to ' s_1 '.

[we use the suffix unity to denote the Quantities belonging to the evolute c]

$$\frac{d\bar{R}}{ds_1} = \frac{d}{ds}[\bar{r} + \lambda\bar{b} + \mu\bar{n}]\frac{ds}{ds_1} \; [\bar{R} = \overline{r_1}]$$

$$\bar{t}_1 = \left[\frac{d\bar{r}}{ds} + \lambda'\bar{b} + \lambda\bar{b}' + \mu'\bar{n} + \mu\bar{n}'\right]\frac{ds}{ds_1}$$

$$\bar{t}_1 = \left[\bar{t} + \lambda'\bar{b} - \tau\bar{n}\lambda + \mu'\bar{n} + \mu(\tau\bar{b} - \kappa\bar{t})\right]\frac{ds}{ds_1}$$

$$= \left[\bar{t} + \lambda'\bar{b} - \tau\lambda\bar{n} + \mu'\tilde{n} + \mu\tau\bar{b} - \kappa\mu\bar{t}\right]\frac{ds}{ds_1}$$

$$\bar{t}_1 = \left[(1 - \mu\kappa)\bar{t} + (\lambda' + \mu\tau)\bar{b} + (\mu' - \tau\lambda)\bar{n}\right]\frac{ds}{ds_1}$$

$\bar{t}_1$ is the tangent to c at Q.

$\therefore$ It is parallel to $\lambda b + \mu\bar{n}$($\because$ $t_1 = \lambda b + \mu\bar{n}$)

Thus we get, coefficients

$$\frac{1-\mu\kappa}{0} = \frac{\mu'-\lambda\tau}{\mu} = \frac{\lambda'+\mu\tau}{\lambda} \qquad \ldots\ldots\ldots\ldots\ldots.(1)$$

$$\therefore 1 - \mu\kappa = 0 \Rightarrow 1 = \mu\kappa \Rightarrow \mu = \frac{1}{\kappa} = \rho \quad \ldots\ldots\ldots..(2)$$

(i.e) $\mu = \rho$

$\therefore$ from (2),

$$\frac{\mu'-\lambda\tau}{\mu} = \frac{\lambda'+\mu\tau}{\lambda}$$

$$\frac{\mu'}{\mu} - \frac{\lambda\tau}{\mu} = \frac{\lambda'}{\lambda} + \frac{\mu\tau}{\lambda}$$

$$\frac{\lambda\mu'-\mu\lambda'}{\lambda\mu} = \frac{\lambda^2\tau+\mu^2\tau}{\lambda\mu}$$

$$\lambda\mu' - \mu\lambda' = \tau[\lambda^2 + \mu^2]$$

$$\Rightarrow \tau = \frac{\lambda\mu' - \mu\lambda'}{\lambda^2 + \mu^2}$$

$$= \frac{\lambda\mu' - \mu\lambda'}{\lambda^2\left[1+\left(\frac{\mu}{\lambda}\right)^2\right]} \qquad \left[\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] \frac{1}{1+(x/y)^2}\frac{d}{dx}\left(\frac{x}{y}\right)\right]$$

(i.e) $\tau = \dfrac{(\lambda\mu' - \mu\lambda')/\lambda^2}{[1+(\mu/\lambda)^2]}$

$(i.e)\ c = \dfrac{d}{ds}[\tan^{-1}(\mu/\lambda)]$

Integrating on both sides,

$\int \tau ds + a = \tan^{-1}\frac{\mu}{\lambda}$ where $a =$ constant.

$\int \tau ds + a = \cot^{-1}[\lambda/\mu]$ (or) $\cot[\int \tau ds + a] = \lambda/\mu$

(i.e) $\mu\cot[\int \tau ds + a] = \lambda$

(ie) $\rho\cot[\int \tau ds + a] = \lambda \quad \ldots\ldots\ldots\ldots..(4) \qquad [\because (2)]$

Sub the values of μ & λ in (*)

$\therefore \bar{R} = \bar{r} + \rho\bar{n} + \rho\cot[\int \tau ds + a]\bar{b} \quad \ldots\ldots\ldots\ldots.(5)$

This is the equation of evolute. where ' a ' = constant

Note:

From equation (5) we get,

The locus of the center of curvature of a space curve is not an evolute.

Example 4:

Show that the involutes of a circular helix are plane curves.

Solution:

We know that the equation of the circular helix is $\bar{r} = (a - \cos u, a\sin u, bu)$ and

$\kappa = \frac{a}{a^2+b^2}, \tau = \frac{b}{a^2+b^2}$

$\therefore \kappa' = 0$ and $\tau' = 0, \tau_1 =$ the torsion of the involute $= 0$

$\Rightarrow$ Involute is a plane curve.

Example 5:

Find the equation of the tangent surface to the curve $\bar{r} = r(u, u^2, u^3) \ldots\ldots\ldots\ldots..(1)$

Solution:

Equation of the tangent surface is,

$\bar{R}(u,s) = \bar{r} + u\bar{t}(s) \quad \ldots\ldots\ldots\ldots..(2)$

Differentiate with respect to ' u ',

$\frac{d\bar{r}}{du} = \dot{\bar{r}} = (1, 2u, 3u^2)$

$\therefore \bar{t} = \frac{\dot{\bar{r}}}{|\dot{\bar{r}}|} = \frac{(1,2u,3u^2)}{\sqrt{1+4u^2+9u^4}}$

Equation of the tangent surface,

$\therefore \bar{R}(u,z) = \bar{r} + u\left[\frac{(1,2u,3u^2)}{\sqrt{1+4u^2+qu^4}}\right]$

$s = |\dot{r}|, \Rightarrow s = \left|\frac{d\bar{\gamma}}{}\right|$

where, $s = |\dot{\bar{r}}|, \Rightarrow S = \left|\frac{d\bar{r}}{du}\right|$

$S = \sqrt{1 + 4u^2 + 9u^4}$

Example 6:

Prove that the locus of cube of curvature is an evolute ⇔ when the curve is plane

Solution:

We know that the equation of the locus of the cube of curvature is,

$\overline{r_1} = \bar{\gamma} + \rho\bar{n}$ … … … …(1) and equation of the evolute is

$\bar{R} = \bar{r} + \rho\bar{n} + \rho\cot\left[\int \tau ds + a\right] b$ … … … … . (2)

Comparing (1) & (2) we get,

$\rho\cot(\psi + a)\bar{b} = 0$ where $\psi = \int \tau ds$

(i.e) $\rho\cot(\psi + a) = 0. (\because \bar{b} \neq 0)$

(i.e) $\cot(\psi + a) = 0\ (\because \rho \neq 0)$

(i.e)

$\psi + a = \cot^{-1} 0$

$\psi + a = \frac{n\pi}{2}$

$\psi = \frac{n\pi}{2} - a$

$\int \tau ds = \frac{n\pi}{2} - a$

Differentiating we get, $\tau = 0$.

Thus locus of cube of curvature is an evolute when $\tau = 0$.

(ie) when the curve is a plane curve.

1.8.Interinsic Equations, Fundamental Existence Theorem for space curves:

If the same curve be referred to a different set of Cartesian axis, then the defining equations are quite different and it is by u_0 means obvious that they refer to the same curve.

Intrinsic Equation (or) Natural Equations:

The intrinsic equations or a curve are of the form, $\kappa = f(s), \tau = g(s)$, which express the curvature and the torsion in terms of the arc length.

Theorem 1: (Uniquiness Theorem for Space Curve)

Let c & c_1 be two curves defined in terms of their respective arc length ' s ' and Let points With. the same values of ' s ' correspond. Then if the curvature and torsion of c have the same values as the curvature & torsion at the corresponding pis of c_1, then c & c_1 ane congruent.

(or)

The curve is uniquely determine except as to position in space when the curvature and torsion are given functions of its arc length.

Let c_1 be moved. So that the two pts on c and c_1 corresponding to $s = 0$ coincide.

Suppose that c_1 is suitably oriented so that the two trials $(\bar{t}, \bar{n}, \bar{b}), (\overline{t_1}, \overline{n_1}, \overline{b_1})$ Coincide at $s = 0$, then, we have $\frac{d}{ds}(t_1, t_1) = \bar{t}' \cdot \bar{t}_1 + \bar{t} \cdot \bar{t}_1'$.

It is possible.

Let it be two curves c and c_1 having equal curvature κ of equal torsion curve τ, For the Same value of s.

Let the suffix unity le west for quantity belonging to c_1.

Now c_1 is moved,

So that the 2 points on c & c_1 corresponding to the some value of coincide.

we have,

$$\frac{d}{ds}(\bar{t} \cdot \bar{t}_1) = \bar{t}' \cdot \bar{t}_1 + \bar{t} \cdot \bar{t}_1'$$

$$= \kappa \bar{n} \cdot \bar{t}_1 + \bar{t} \cdot \kappa_1 \bar{n}_1$$

$$= \kappa \bar{n} \bar{t}_1 + \bar{t} \cdot \kappa \bar{n}_1 \ (\because \kappa = \kappa_1)$$

$$\frac{d}{ds}(\bar{t}.\bar{t}_1) = \kappa[\bar{n}.\bar{t}_1 + \bar{t}.\bar{n}_1]$$

$$\frac{d}{ds}(\bar{n}.\bar{n}_1) = \bar{n}' \cdot \bar{n}_1 + \bar{n} \cdot \bar{n}_1'$$

$$= (\tau \bar{b} - \kappa \bar{t}) \cdot \bar{n}_1 + \bar{\bar{n}} \cdot (\tau_1 \bar{b}_1 - \kappa_1 \bar{t}_1)$$

$$= (\tau \bar{b} - \kappa \bar{t})\bar{n}_1 + \bar{n}(\tau \overline{b_1} - \kappa_1 \overline{t_1})$$

$\frac{d}{d}(\bar{n}\cdot\bar{n}_1) = [\tau(\bar{b}\bar{n}_1 + \overline{b_1}\bar{n}) - \kappa(\bar{t}\bar{n}_1 + \bar{t}_1.\bar{n})]. [\because \tau = \tau_1 \text{ and } \kappa = \kappa_1]$

Similarly, $\frac{d}{ds}(\bar{b}\cdot\bar{b}_1) = (-\tau\bar{n})\bar{b}_1 + \bar{b}\cdot(-\tau\cdot\bar{n}_1)$

$\frac{d}{ds}(\bar{b},\bar{b}_1) = -\tau\bar{n}\bar{b}_1 + \tau\bar{b}\bar{n}_1$

$= -\tau(\bar{n}\bar{b}_1 + \bar{b}\bar{n}_1)\ (\because \tau = \tau_1)$

$$\therefore \frac{d}{ds}[\bar{t}\cdot\overline{t_1} + \bar{n}\cdot\overline{n_1} + \bar{b}\cdot\bar{b}_1] = \kappa[\bar{n}\cdot\overline{t_1} + \bar{t}\cdot\overline{n_1}] + \tau[\bar{b}\cdot\bar{n}_1 + \bar{b}_1\cdot\bar{n}]$$
$$-\kappa[\bar{t},\bar{n}_1 + E_1,\bar{n}] - \tau[\bar{n}\cdot\overline{b_1} + \bar{b}\cdot\bar{n}_1]$$

$\frac{d}{ds}[\bar{t}\cdot\bar{t}_1 + \bar{n}\cdot\bar{n}_1 + \bar{b}\cdot\bar{b}_1] = 0.$

$\Rightarrow \bar{t}\cdot\bar{t}_1 + \bar{n}\cdot\bar{n}_1 + \bar{b}\cdot\bar{b}_1 = \text{ constant}$ ....(1)

we move c_1 in such away that at $s = 0$, The two triads $(\bar{t},\bar{n},\bar{b})\&(\bar{t}_1,\bar{n}_1,\bar{b}_1)$ coincide then,

$$\left.\begin{aligned} \bar{t}_1.\bar{t} &= \cos 0 = 1 \\ \bar{n}_1\cdot\bar{n} &= \cos 0 = 1 \\ \bar{b}_1.\bar{b} &= \cos 0 = 1 \end{aligned}\right\} \text{at } s = 0.$$

$\therefore (1) \Rightarrow \bar{t}\cdot\bar{t}_1 + \bar{n}\cdot\bar{n}_1 + \bar{b}\cdot\bar{b}_1 = 3$ (when $s = 0$).

But the sum of the 3 cosines is equal to ' s ' If angle in zero (or) an integral multiple of 2π (ie)

$\bar{t} = \bar{t}_1, \bar{n} = \bar{n}_1, \bar{b} = \bar{b}_1$

Thus gives $\bar{r}' = \bar{r}_1'$

(ie) $\frac{d}{ds}(\bar{r} - \bar{r}_1) = 0$

$\Rightarrow \bar{r} - \bar{r}_1 = \text{ constant.}$

But at $s = 0, \bar{r} = \bar{r}_1$

$\therefore \bar{r} = \bar{r}_1$ at all the corresponding points.

Hence the two curves coincides

Theorem 2:

Fundamental Existence Theorem for Space curves.

If $\kappa(s), \tau(s)$ are cts, *functions* of the real variable 's', where $s \geq 0$, then there exists a Space curve for which κ is the curvature, τ is the torsion, and 's' in the arc length measured from some suitable base point Such a curve is uniquely determined to within a Eudidecen motion.

Proof :

Using the given functions $\kappa(s)$ & $\tau(s)$. we form the following differential equations,

$$\left.\begin{aligned}\frac{d\alpha}{ds} &= \kappa\beta, \\ \frac{d\beta}{ds} &= \tau\gamma - \kappa\alpha, \\ \frac{d\alpha}{ds} &= -\tau\beta\end{aligned}\right\} \text{..........(1)}$$

We know that the system (1) has a unique solution.

$(\alpha_0, \beta_0, \nu_0)$ for any given initial condition $s = s_0 = 0$

In particular for the given initial condition $\alpha(s) = 1, \beta(S) = 0, \gamma(S) = 0$ at $s = 0$.

(1) has a unique solution.

Denoted it by $\alpha_1, \beta_1, \gamma_1$

Similarly for the initial condition,

$(s) = 1, \beta(S) = 0, \gamma(S) = 0$ at $s = 0$

Let the solution of (1) be $\alpha_2, \beta_2, \gamma_2$.

Again for the initial condition,

$\alpha(s) = 0, \beta(s) = 0, \gamma(s) = 1$ at $s = 0$.

To prove $that$: $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1$$

$$\alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1$$

Consider, $\frac{d}{ds}[\alpha_1^2 + \beta_1^2 + \gamma_1^2] = 2[\alpha_1\alpha_1^1 + \beta_1\beta_1' + \gamma_1\gamma_1^1]$

$$= 2[\alpha_1(\kappa\beta_1) + \beta_1(\tau\gamma_1 - \gamma\alpha_1) + \alpha_1(-\tau\beta_1)]\ [\because (1)]$$

$= 2[\alpha_1\gamma\beta_1 + \beta_1\gamma_1\tau - \beta_1\kappa\alpha_1 - \alpha_1\beta_1\tau]$

$= 0.$

$\Rightarrow \alpha_1^2 + \beta_1^2 + \gamma_1^2 =$ constant ....(2)

But at $s = 0, \alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0$.

Substitute in (2).

(2) $\Rightarrow 1 + 0 + 0 =$ constant $\Rightarrow$ constant $= 1$.

$$\left.\begin{aligned}\Rightarrow \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \\ \text{Similarly,}\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1 \\ \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1\end{aligned}\right\} \dots\dots\dots\dots(3)$$

$$\text{Similarly,}\quad \left.\begin{aligned}\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 &= 0 \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0 \\ \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 &= 0\end{aligned}\right\} \dots\dots\dots\dots..(4)$$

$A = \begin{bmatrix}\alpha_1 & \beta_1 & \nu_1 \\ \alpha_2 & \beta_2 & \nu_2 \\ \alpha_3 & \beta_3 & \nu_3\end{bmatrix}$ is orthogonal.

(ie) $AA' = I$

Where $A' =$ The transpose of A.

Since the columns of A are Linearly independent.

$\Rightarrow A$ is non-singulas.

(ie) A^{-1} exists

$\therefore$ pre-multiply (5) by A^{-1}.

$(5) \Rightarrow A^{-1}AA' = A^{-1}I.$

$\Rightarrow A' = A^{-1}I.$

$\therefore AA^{-1} = I$

(i.e) $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_1 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left.\begin{aligned} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 &= 1 \\ \beta_1^2 + \beta_2^2 + \beta_3^2 &= 1 \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 &= 0 \\ \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 &= 0 \\ \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 &= 0 \end{aligned}\right] \quad \ldots\ldots\ldots\ldots\ldots(6)$$

It follows that there are three mutually orthogonal unit vectors

put $\bar{t} = (\alpha_1, \alpha_2, \alpha_3)$

$\bar{n} = (\beta_1, \beta_2, \beta_3)$

$\bar{b} = (x_1, x_2, x_3)$

The relation (6) show that the 3-values

$\bar{t}, \bar{n}, \bar{b}$ are unit vectors, and they are 3 mutually perpendicular vectors.

$$\bar{r} = \int_0^s \bar{t}ds$$

Then $\bar{r} = \bar{r}(s)$ is the position vector of a point on a curve which has,

$\bar{t}$ as tangent vector

$\bar{n}$ as principal normal.

$\bar{b}$ as binormal,

κ as curvature,

τ as torsion.

s as arc length

This proves the existence of the required cure.

Example 1:

Show that the intrinsic equations of the curve given by

$x = ae^u \cos u, y = ae^u \sin u, z = be^u are$

$\kappa = \frac{\sqrt{2}a}{(2a^2+b^2)^{1/2}} \cdot \frac{1}{s}, \tau = \frac{b}{(2a^2+b^2)^{1/2}} \cdot \frac{1}{s}.$

Proof:

Given $x = ae^u \cos u$

$\Rightarrow \dot{x} = ae^u \cos u + ae^u(-\sin u)$

$\Rightarrow \dot{x} = ae^u[\cos u - \sin u]$

$given, y = ae^u \sin u$

$\Rightarrow \dot{y} = ae^u \sin u + ae^u \cos u$

$\Rightarrow \dot{y} = ae^u[\sin u + \cos u]$

given, $z_\gamma = be^u$

$\therefore \bar{r} = (ae^u \cos u, \text{a } e^u \sin u, \text{be } u)$

$\dot{\bar{r}} = (ae^u[\cos u - \sin u], ae^u[\sin u + \cos u], \text{be } u)$

$|\dot{\bar{r}}| = \sqrt{a^2(e^u)^2[\cos u - \sin u]^2 + a^2(e^u)^2[\sin u + \cos u]^2 + b^2(e^u)^2}$

$= e^u\sqrt{a^2[\cos^2 u + \sin^2 u - 2\cos u\sin u + \sin^2 u + \cos^2 u + 2\text{cog}[u\sin u] + b^2}$

$= e^u\sqrt{a^2[2(\cos^2 u + \sin^2 u)] + b^2}$

$|\dot{\bar{r}}| = e^u\sqrt{2a^2+b^2}$

$\therefore s = \left[e^u\sqrt{2a^2+b^2}\right].$

$s = \int_{-\infty}^{u} e^u\left(\sqrt{2a^2+b^2}\right)du$

$= \sqrt{2a^2+b^2}\int_{-\infty}^{u} e^u du$

$= \sqrt{2a^2+b^2}\,e^u = \dot{s}$

$s = \dot{s}$ .... (1)

$\bar{r}' = \frac{\dot{\bar{r}}}{|\dot{\bar{r}}|} = \frac{(ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u)}{e^u\sqrt{2a^2+b^2}}$

$\bar{r}' = \frac{e^u(a(\cos u - \sin u), a(\cos u + \sin u), b)}{e^u\sqrt{2a^2+b^2}}$

$\bar{r}' = \frac{(a(\cos u - \sin u), a(\cos u + \sin u), b)}{\sqrt{2a^2+b^2}}$ .... (1)

$\therefore \bar{r}'' = \left[\frac{(a(-\sin u - \cos u), a(-\sin u + \cos u), 0)}{\sqrt{2a^2+b^2}}\right]\frac{du}{ds}$

$\kappa\bar{n} = \frac{(-a(\sin u + \cos u), a(\cos u - \sin u), 0)\frac{1}{s}}{\sqrt{2a^2+b^2}}$

Taking modulus on both sides and squaring,

$$\kappa = \frac{\sqrt{2}a}{\sqrt{2a^2+b^2}} \cdot \frac{1}{s} \left[s = \dot{s} = \frac{ds}{du} \; implies \frac{du}{ds} = \frac{1}{\dot{s}} = \frac{1}{s}\right]$$

$$\therefore (3) \Rightarrow s\bar{r}'' = \frac{1}{\sqrt{2a^2+b^2}}(-a(\sin u + \cos u), a(-\sin u + \cos), 0) \ldots\ldots\ldots..(4)$$

$$s^2\bar{r}''' + \bar{r}''s = \frac{1}{\sqrt{2a^2+b^2}}(-a(\cos u - \sin u), (-a(\sin u + \cos u), 0) \;\ldots\ldots\ldots..(5)$$

$$s^3 \cdot (\bar{r}'' \times \bar{r}''') = \frac{1}{2a^2+b^2}(0,0,2a^2) \quad \ldots\ldots\ldots\ldots\ldots.(B)$$

$$\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ -a(\cos u + \sin u) & a(\cos u - \sin u) & 0 \\ -a(\cos u - \sin u) & -a(\cos u + \sin u) & 0 \end{vmatrix}$$

$$= \bar{k}[a^2(\cos u + \sin u)^2 + a^2(\cos u - \sin u)^2]$$

$$= \bar{k}(a^2 2)$$

Taking scalar product of (2) & (B),

$$S^3[\bar{r}', \bar{r}'', \bar{r}'''] = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$$

$$S^3(\kappa^2\tau) = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$$

$$\frac{S^3 2a^2\tau}{(2a^2+b^2)S^2} = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$$

$$\Rightarrow \tau = \frac{b}{S(2a^2+b^2)^{1/2}}$$

1.9.Helices:

Cylindrical Helix:

A cylindrical helix is a space curve which lies on a cylinder and cuts the generators at a constant angle.

Its tangent makes a constant angle ' α ' with a fixed line known as the axis of the helix.

Note:

1. Helices more general than cylindrical helices.
2. Helix mean cylindrical helix. (in this book).

A characteristic property of Helices:

The ratio of the curvature to the torsion is constant at all points.

(ie) $\frac{k}{\tau}$ = constant.

Proof:

Let $\bar{a}$ = a unit vector in the direction of the axis of the cylinder.

then, $\bar{t} \cdot \bar{a} = \cos\alpha$ ……………(1)

Where α = constant angle.

Differentiate with respect to ‘s’,

$\bar{t}' \cdot \bar{a} + \bar{t}(0) = 0.$

$k\bar{n} \cdot \bar{a} = 0$

$\Rightarrow \bar{n} \cdot \bar{a} = 0 (\because k \neq 0) \Rightarrow \bar{n}$ is perpendicular to $\bar{a}$.

This shows that the principal normal is everywhere perpendicular to $\bar{a}$ the generator But principal normal is everywhere perpendicular to the rectrifying plane,

$\therefore \bar{a} = \lambda\bar{t} + \mu\bar{b}$ ……………(3)

(3), $\bar{t} \Rightarrow \bar{a} \cdot \bar{t} = \lambda\bar{t} \cdot \bar{t} + \mu\bar{b} \cdot \bar{t}$

$\Rightarrow \cos\alpha = \lambda$

(3) $\cdot \bar{b} \Rightarrow \bar{a} \cdot \bar{b} = \lambda\bar{t} \cdot \bar{b} + \mu\bar{b} \cdot \bar{b}$

$\Rightarrow \sin\alpha = 0 + \mu$

$\Rightarrow \sin\alpha = \mu$

$\therefore (3) \Rightarrow \bar{a} = \cos\alpha\bar{t} + \sin\alpha\bar{b}$

Differentiate with respect to 's',

$\bar{a}' = \cos\alpha\bar{t}' + \sin\alpha\bar{b}'$

$0 = \cos\alpha(\kappa\bar{n}) + \sin\alpha(-\tau\bar{n})$

$0 = (\kappa\cos\alpha - \tau\sin\alpha)\bar{n}.$

$\Rightarrow 0 = \kappa\cos\alpha - \tau\sin\alpha \; [\because \bar{n} \neq 0]$

$\Rightarrow \kappa\cos\alpha = \tau\sin\alpha$

$\Rightarrow \frac{\kappa}{\tau} = \frac{\sin\alpha}{\cos\alpha}.$

$\Rightarrow \frac{\kappa}{\tau} = \tan\alpha = \text{constant}$

Remark:

Converse is also true.

(ie) If $\frac{\kappa}{\tau}$ = constant for a curve then it should be a helix.

Proof:

Given $\frac{k}{\tau}$ = constant = c (say)

$\kappa = c\tau$

We know that

$\bar{t}' = \kappa\bar{n}$

$= c\tau\bar{n}$

$= -c\bar{b}' [\therefore \bar{b}' = -\tau\bar{n}]$

$\Rightarrow \bar{t}' + c\bar{b}' = 0$

$\Rightarrow \frac{d\bar{t}}{ds} + c\frac{d\bar{b}}{ds} = 0$

$\Rightarrow \frac{d}{ds}[\bar{t}+c\bar{b}]=0$

$\Rightarrow \bar{t}+c\bar{b}=$ constant $=\bar{a}($ say $)$

Taking scalar product of each side with $\bar{t}$

$\therefore \bar{t}\cdot\bar{t}+c\bar{b}\cdot\bar{t}=\bar{a}\cdot\bar{t}$

$1+0=\bar{a}\cdot\bar{t}$

$\Rightarrow \bar{a}\cdot\bar{\bar{t}}=1=$ constant

(i.e) the tangent at every point of the curve makes a constant angle with a fixed vector Q. $\therefore$ The curve is a helix

Circular Helix:

A circular helix is one which lies on the surface of a circular cylinder, the axis of the helix being that of the cylinder.

It the z-axis is the axis of the helix, the parametric eqn. of the curve is

$x=a\cos u, y=a\sin u, z=bu$, where $a>0$

$\therefore \bar{r}=(a\cos u, a\sin u, bu)$

If $b>0$, then the helix is right hand.

If $b<0$, then the helix is left hand.

In circular helix, both κ & τ are constant $\Rightarrow \frac{\kappa}{\tau}$ is constant.

$\left[\frac{1}{\kappa}=\rho=a\sec^2\alpha, \frac{1}{\tau}=\sigma=a\text{cosec}\alpha\sec\alpha\right]$.

Note:

The pitch of the helix $=2\pi b=$ The displacement along the axis corresponding to a complete turn round the axis.

Example 1:

Prove that the helix at the constant curvature is necessary a circular helix.

(or)

For any general helix ' c ' there is a simple relation bet' n its curvature & that of the plane curve c_1 obtained by projecting on a plane orthogonal to its axis.

Proof:

Let ' c ' be a general helix and ' c_1 ' be the curve obtained by projecting' ' c '. on a plane orthogonal to x-axis.

To prove that: The projection is a circular & hence the helix is a circular helix we use the suffix unity to denote the entire belonging to 'c_1'

Let ' P ' be a point on c and ' Q ' be a corresponding pt of C_1.'

$\overline{OP} = \overline{OQ} + \overline{QP}$

$\bar{r} = \bar{r}_1 + (\bar{r}.\bar{a})\bar{a}$

Differentiate with respect to 's',

$\frac{d\bar{r}}{ds} = \frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} + \left[\frac{d\bar{r}}{ds} \cdot \bar{a}\right] \bar{a}$

$(ie)\bar{t} = \bar{t}_1 \cdot \frac{ds_1}{ds} + [\bar{t} \cdot \bar{a}]\bar{a}$

$= \bar{t}_1 \cdot \frac{ds_1}{ds} + (\cos\alpha)\bar{a} \quad \dots\dots\dots\dots (1)$

$\therefore \bar{t}.\bar{\mathrm{t}}_1 = \left(\bar{t}_1 \cdot \frac{ds_1}{ds} + \cos\alpha\bar{a}\right) \cdot \left(\overline{t_1} \cdot \frac{ds_1}{ds} + \cos\alpha\bar{a}\right)$

$= \bar{\mathrm{t}}_1 \cdot \bar{t}_1 \left(\frac{ds_1}{ds}\right)^2 + (\bar{a} \cdot \bar{a}) \cos^2\alpha \; [\because \bar{t} \cdot \bar{a} = 0]]$

$1 = \left(\frac{ds_1}{ds}\right)^2 + \cos^2\alpha$

$\Rightarrow 1 - \cos^2\alpha = \left(\frac{ds_1}{ds}\right)^2$

$\Rightarrow \sin^2\alpha = \left(\frac{ds_1}{ds}\right)^2$

$\Rightarrow \frac{ds_1}{ds} = \sin\alpha \quad \dots\dots\dots\dots..(2)$

Substituting in (1),

$\bar{t} = \bar{t}_1 \cdot \sin\alpha + \bar{a}\cos\alpha$

Differentiate with respect to 's'.

$\frac{dt}{ds} = \frac{dt_1}{ds_1}\sin\alpha + 0$

$\frac{dt}{ds} = \frac{dt_1}{ds_1} \cdot \frac{ds_1}{ds} \cdot \sin\alpha$

$\bar{t}' = \bar{t}_1' \cdot \frac{ds_1}{ds}\sin\alpha \qquad [\because (2)]$

$\kappa\bar{n}' = \kappa_1\bar{n}_1(\sin\alpha)(\sin\alpha)$

$\kappa\bar{n} = \kappa_1\bar{n}_1\sin^2\alpha.$

$\bar{n}$ is parallel to $\bar{n}_1$ and $\kappa = \kappa_1 \sin^2\alpha$

Given that helix ' c ' has a constant curvature κ

$\therefore \kappa_1\sin^2\alpha = \kappa =$ is also constant

$\Rightarrow \kappa_1$ is also constant.

Thus the plane curve c_1 is such that its curvature κ_1 is constant

$\therefore c_1$ is a circle

Thus c is a circular helix

Example:

Definition: Spherical Indicatrices

[The locus of a point whose position vector in the tangent vector $\bar{t}$ to a curve γ is called the

Spherical Indicatrisc of the tangent to γ.]

Prove that the tangent to the indicatrix is parallel to the principal normal at the corresponding point of γ. Show that the curvature κ_1 and the torsion τ_1 of the indicatrix are given by,

$$\kappa_1^2 = \frac{(\kappa^2+\tau^2)}{\kappa^2}, \tau_1 = \frac{(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2+\tau^2)}$$

Proof:

From the definition of the Spherical indicatrix.

We note that "Let ' 0 ' be the center of the unit sphere. Let us dram "o' the unit tangent vectors If the different points of γ in the positives direction of $\bar{t}$ then the curve traced on the unit sphere by the extrenities of the unit tangent through ' O ' is the spherical indicatrix".

Then, $\bar{r}_1 = \bar{t}$

Differentiate with respect to ' s_1 '.

$$\frac{d\overline{r_1}}{ds_1} = \frac{dt}{ds_1}$$

(i.e) $\bar{t}_1 = \frac{d\bar{t}}{ds} \cdot \frac{ds}{ds_1}$

$$\Rightarrow \bar{t}_1 = \bar{t}' \cdot \frac{ds}{ds_1}$$

$$\Rightarrow \bar{t}_1 = \kappa\bar{n}\frac{ds}{ds_1} \left(\because \frac{ds_1}{ds} = \kappa\right)$$

$$\Rightarrow \bar{t}_1 = \bar{n} \quad \ldots\ldots\ldots(1) \left(\therefore \frac{ds_1}{ds} = \kappa\right) \quad \ldots\ldots\ldots(2)$$

$\therefore$ (1) $\Rightarrow$ The tangent to spherical indicatrix is parallel to $\bar{n}$ of γ.

To find: κ_1

Differentiate (1) with respect to $'s_1'$,

$$\frac{dt_1}{ds_1} = \frac{d}{ds_1}(\bar{n})$$

$$\kappa_1\bar{n}_1 = \frac{d}{ds}(\bar{n})\left(\frac{ds}{ds_1}\right)$$

$$\kappa_1\bar{n}_1 = (\tau\bar{b} - \kappa\bar{t}).\left(\frac{1}{\kappa}\right)$$

Taking dot product of (3) with itself,

$$(\kappa_1, \bar{n}) \cdot (\kappa_1, \bar{n}) = \frac{1}{\kappa^2}(\tau\bar{b} - \kappa\bar{t})(\tau\bar{b} - \kappa\bar{t})$$

$$\Rightarrow \kappa_1^2 = \frac{1}{\kappa^2}(\tau^2 + \kappa^2)$$

To find τ_1 :

$(1) \times (3) \Rightarrow$

$\bar{t}_1 \times \kappa_1 \bar{n}_1 = \bar{n} \times \left[\frac{\tau\bar{b} - k\bar{t}}{\kappa}\right]$

$\kappa_1 \bar{b}_1 = \frac{\tau\bar{t} + \kappa\bar{b}}{\kappa}$

$\kappa\kappa_1 \bar{b}_1 = \tau\bar{t} + \kappa\bar{b}$

Differentiate with respect to $'s_1'$,

$\frac{d}{ds_1}[\kappa\kappa_1]\bar{\mathrm{b}} + \kappa\kappa_1 \mathbf{\bar{b}_1}' = \frac{\mathrm{d}}{\mathrm{d}s_1}[\tau\bar{t} + \kappa\bar{b}]$

$\frac{d}{ds_1}[\kappa\kappa_1]\bar{\mathrm{b}} + \kappa\kappa_1(-\tau_1\bar{n}_1) = [\tau'\bar{t} + \tau\bar{t}' + \kappa'\bar{b} + \kappa\bar{b}']\left(\frac{ds}{ds_1}\right)$

$\frac{d}{ds_1}[\kappa\kappa_1]\bar{\mathrm{b}} - \tau_1\bar{n}_1\kappa\kappa_1 = [\tau'\bar{t} + \tau\kappa\bar{n} + \kappa'\bar{b} - \kappa\tau\bar{n}]\left(\frac{ds}{ds_1}\right)$

$\frac{d}{ds_1}[\kappa\kappa_1]\bar{\mathrm{b}} - \tau_1\bar{n}_1\kappa\kappa_1 = [\tau'\bar{t} + \kappa'\bar{b}]$ ……………. (4) $\left[\because \frac{ds}{ds_1} = \kappa\right]$

Taking dot product of (3) with (4)

$-\tau_1\kappa_1^2\kappa = \frac{(\tau\bar{b} - \kappa\bar{t}\prime)}{\kappa} \cdot \left(\frac{\tau'\bar{t} + \kappa'\bar{b}}{\kappa}\right)$

$= \frac{1}{\kappa^2}(\tau\kappa' - \kappa'\tau')$

$\tau_1 = \frac{\kappa\tau' - \tau\kappa'}{\kappa^3\kappa_1^2}$

$\tau_1 = \frac{[\kappa\tau' - \kappa'\tau]\kappa^2}{\kappa^3(\kappa^2 + \tau^2)} \quad \because \kappa_1^2 = \frac{\tau^2 + \kappa^2}{\kappa^2}$

$\therefore \tau_1 = \frac{(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2 + \tau^2)}$

Example 1:

The locus of a point whose position vector in the binormal $\bar{b}$ of a curve γ is called the spherical Indicatrix of the Binormal to γ.

Prove that its curvature κ_2 & torsion τ_2 are g^n. by

$$\kappa_2^2 = \frac{(\kappa^2 + \tau^2)}{\tau^2},$$

$$\tau_2 = \frac{(\tau\kappa_1' - \kappa\tau')}{\tau(\kappa^2 + \tau^2)}$$

Proof:

Let γ be the given curve with equation,

$\bar{r} = \bar{r}(s)$.

Let $\bar{r}_2$ be the position vector an any p^+. on the Spherical indicatrix.

then $\bar{r}_2 = \bar{b}$.

Diff wi. to ' s_2 '

$\frac{d\bar{r}_2}{ds_2} = \frac{d\bar{b}}{ds_2}$

(i.e) $\overline{t_2} = \frac{db}{ds} \cdot \frac{ds}{ds_2}$

(ie) $\overline{t_2} = -\tau\bar{n}\frac{ds}{ds_2}$

$\Rightarrow \bar{t}_2 = -\bar{n}$ ……….(1)

$\frac{ds_2}{ds} = \tau$ …………(2)

(1) ⇒ the tangent to spherical indicatrix is parallel to $\bar{n}$ of γ ,

To find κ_1:

Differentiate (1) with respect to ' s_2' ',

$\frac{dt_2}{ds_2} = \frac{d}{ds_2}[-\bar{n}]$

$= \frac{d}{ds}[-\bar{n}]\frac{ds}{ds_2}[\because by\ (2)]$

(i.e) $k_2\overline{n_2} = (-\tau\bar{b} + \kappa t)\frac{1}{\tau}$ ………….(3)$[\because \bar{n}' = \tau\bar{b} - \kappa\bar{t}]$

Taking dot product of (3) with itself,

$\kappa_2^2 = \frac{\tau^2+\kappa^2}{\tau^2}$

To find τ_2:

(1) $\times$ (3) $\Rightarrow \bar{t}_2 \times \kappa_2 n_2 = (-\bar{n}) \times \left[\frac{\tau\bar{b}+\kappa\bar{t}}{\tau}\right]$

$\kappa_2\bar{b}_2 = \frac{\tau\bar{t}+\kappa\bar{b}}{\tau}$

(or) $\tau\kappa\bar{b}_2 = \tau\bar{t} + \kappa\bar{b}_1$.

Differentiate with respect to ' S_2' '.

$\frac{d}{ds_2}[\tau\kappa_2\bar{b}_2] = \frac{d}{ds_2}[\tau\bar{t} + \kappa\bar{b}]$

$\frac{d}{ds_2}[\tau\kappa_2]\bar{b}_2 + \tau\kappa_2\bar{b}_2' = [\tau'^{\bar{t}} + \tau\bar{t}_1'' + \bar{b}k' + \kappa\bar{b}']\frac{ds}{ds_2}$

$= [\tau'\bar{t} + \tau\kappa\bar{n} + \kappa'\bar{b} + \kappa(-\tau\bar{n})]\frac{ds}{ds_2}$

UNIT II

Intrinsic properties of a surface: Definition of a surface – curves on a surface – Surface of revolution – Helicoids – Metric- Direction coefficients – families of curves- Isometric correspondence- Intrinsic properties.

Chapter 2: Sections 2.1 – 2.9

2. Intrinsic properties of a surface

2.1.Definition of a Surface:

A surface is a locus of a point' P 'which satisfies a relation or the form

$F(x,y,z)=0$ ………. (1)

This equation is called the implicit or constrained equation of the surfaces.

An explicit form in which the coordinate of a point on the surface are expressed interns of two parameters.

The parametric or freedom equation of a surface take the form.

$x=f(u,v), y=g(u,v).$

$z=h(u,v)$ …………… (2)

Where u and v are parameters, where k is real value on the vary freely in some domain D.

The functions g, f, h are single values and continuous to passes continuous partial derivatives of r[th] order. In this case, the surface is said to be of class r. parameters Such as u,v are frequently called linear co-ordinates, The point determined by the pair (u,v) is referred as a point (u,v) itself.

When the parametric equation of the surface is given, we will find suitable constrained equations for example.

Consider a surface given by the parametric equation

$$\left.\begin{array}{l} x=u+v \\ y=u-v, z=4uv \end{array}\right\}$$ … … … (3) where u and v take real values.

we see that $x^2-y^2=(u+v)^2-(u-v)^2=4uvt$

$z=4uz$ ………. (4)

Which represents a certain hyperbolic parabolic.

The parametric equation are not unique

Example 1:

$x=u, y=v, z=u^2-v^2$ … … … … (5) represents the same equation. Sometimes the constraint equations obtain by eliminating the parameters represents more than one Surface.

For example:- Consider the surface equations, given by

$$\left.\begin{aligned} x &= u \cosh v \\ y &= u \sinh v \\ z &= u^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots(6) \; where\; u \; and \; v \text{ are real numbers}$$

we have $x^2 - y^2 = u^2(\cosh^2 v - \sinh^2 v) = u^2 = \tau.$

$x^2 - y^2 = z$

Which again represents equations of parabolic hyperbola

Two representation of same surface such as,

$$\begin{aligned} x &= u + v \\ y &= u - v \end{aligned} \;\; \& \;\; x = u$$

$z = 4uv \qquad z = u^2 - v^2$ are related by parametric transformation of the form.

$$\left.\begin{aligned} u' &= \phi(u,v) \\ v' &= \psi(u,v) \end{aligned}\right\} \ldots\ldots\ldots(7)$$

In certain domain D,

This transformation is said to be proper in ϕ & ψ are single valued & hence they have non-Vanishing Jacobians.

(i.e.) $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in D …………(8)

The position vector $\vec{r} = (x, y, z)$ of a point on the Surface is a funs $u \& v$ with the Same continuity and differentiability property here partial differentiation with respect to u and v will be denoted by suffixes, $\left.\begin{aligned} \vec{r_1} &= \frac{\partial \vec{r}}{\partial u'} \\ \vec{r_2} &= \frac{\partial r}{\partial v'} \end{aligned}\right\} \ldots\ldots\ldots(9)$

Definition :

An ordinary point is defined as 1 for which $r_1 \times r_2 \neq 0$

(i.e.) $\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 2.$

Show that the property of being an ordinary point is unaltered by a proper parametric transformation.

Solution:

$$u' = \phi(u,v)$$

$$v' = \psi(u,v)$$

(i.e.) $r_1 \times r_2 \neq 0.$

$$\frac{d\vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \neq 0.$$

By the Jacobian property $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ and $\frac{\partial\vec{r}}{\partial u'} \times \frac{\partial\vec{r}}{\partial v'} \neq 0$.

⇒ The ordinary point is unaltered by a proper parametric Transformation.

Note:

A point which is not an ordinary points called a point of singularity.

The two parametric curves through the point P are orthogonal if $\vec{r_1} \cdot \vec{r_2} = 0$ at the point P.

If this condition is satisfied at every point, then the two system of parametric curve are orthogonal.

2.2. Curves on Surface:

Let us consider a surface r=r(u, v) defined on a domain D and if u and v are functions at single parameter 't' then the position vector r becomes function of single parameter t and hence it is locus is a curve lying on a surface r=r(u, v).

Let u=U(t), v=V(t) then r=r(U(t), V(t)) is a curve lying on a surface in D. The equation u=U(t) and v=V(t) are called the curvilinear of the curve on the surface.

Parametric curves: Let r=r (u, v) be the equation of the surface defined on a domain D.

Now by keeping u=constant (or) v=constant, we get the curves of special importance and are called the parametric curves. Thus if v=c(say) then as u varies then the point r=r(u, c) describe a parametric curves called u-curve. For u-curve, u is a parameter and determine a sense along the curve.

The tangent to the curve in the sense of u increasing is along the vector. Similarly, the tangent to v-curve in the sense v increasing is along the vector. We have two system of parametric curves viz. u-curve and v-curve and since we know that 0 The parametric curve of different systems can't touch each other. If =0 at a point p, then two parametric curves through the point p are orthogonal. If this condition is satisfied at every point.

(i.e.) For all values of u and v in the domain D, the two system of parametric curves are orthogonal. Tangent plane: Let the equation of the curve be u=u(t), v=v(t) then the tangent is parallel to the vector $\dot{r}$ where

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial u}\frac{du}{dt} + \frac{\partial r}{\partial v}\frac{dv}{dt}$$

$$= r_1\frac{du}{dt} + r_2\frac{dv}{dt}$$

$$\Longrightarrow dr = r_1 du + r_2 dv$$

But r_1 and r_2 are non-zero and independent vectors.

The tangent to the curve through a point p on the surface lie in the plane. This plane is called the tangent plane at p.

Tangent line to the surface: Tangent to the any curve drawn on a surface is called a tangent line to the surface.

Definition:

The normal to the surface at p is a line through p and perpendicular to the tangent plane at p. Since r_1 and r_2 lie in the tangent plane at p and passes through $p_1$the normal is perpendicular to both r_1 and r_2 and it is parallel to $r_1 \times r_2$. The normal at p is fixed by the following convention.

If N denotes the unit normal vector at p, then r_1, r_2 and N should form convention, a right handed system using this convention, we get

$$N = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H} \; where \; H = |r_1 \times r_2|$$

Since $r_1 \times r_2 \neq 0$, we have $H = |r_1 \times r_2| \neq 0$

$\Longrightarrow NH = r_1 \times r_2.$

2.3.Surface of Revolution:

The Sphere:

Obtain the equation of a sphere and a general surface of revolution about z-axis. When the polar angles that is the colatitude u and longitude v are take as parameter on a sphere at center o. radius a, The of any point is given by,

$\vec{r} = (a \sin u \cos v, a\sin u\sin v, a\cos u)$ ………….. (1)

and here the poles $u = 0$ & $u = \pi$ are the Singularities and domain of $u \cdot v$ is $0 < u < \pi$ and $0 < u \leq 2\pi$.

The parametric curves $v =$ constant ave the meridians and $u =$ constant are the parallel and the two systems ane orthogonal.

$$r_1 = (g'\cos v, g'\sin v, f')$$
$$r_2 = (-g\sin v, g\cos v, 0)$$

For:

$$\vec{r_1} = (a\cos u\cos v, a \sin v \cos u, -a\sin u)$$
$$\vec{r_2} = (-a\sin u\sin v, a\sin u\cos v, 0)$$
$$\vec{r_1} \cdot \vec{r_2} = (-a^2 \sin u \cos u \sin v \cos v + a^2 \sin u \cos u \sin v \cos v)$$
$$= 0$$

$$\vec{r_1}^{\,2} + \vec{r}_2^{\,2} = a^2\cos^2 u\cos^2 v + a^2\sin^2 v\cos^2 u + a^2\sin^2 u + a^2\sin^2 u\sin^2 v + a^2\sin^2 u\cos^2 v$$

[$= a^2\cos^2 u(1) + a^2\sin^2(1) + a^2\sin^2$

$= a^2(1) + a^2\sin^2 u$

$= a[1 + \sin^2 u)]$

This show that the Normal, $\vec{N}$ is directed outwards from the sphere.

The General Surface of Revolution:-

Taking z-axis for the axis of revolution, let the generating curve in the xoz plane is given by the parametric equations $x = g(u), y = 0,$

$z = f(u)$

If v, u the angle of rotation about z axis the position vector u, v, is given by,

$\vec{\gamma} = (g(u)\cos v, g(u)\sin v, f(u))$

The domain of (u, v) is $0 \le v \le 2\pi$ as in the case of the sphere $v =$ constant are medians given by the various positions of the generating curve & $u =$ constant are the parallels.

(i.e.,) Circular planes parallel to xoy plane the respective vector $\vec{r_1}$ and $\vec{r_2}$ are

$\vec{r_1} = (g'\cos v, g'\sin v, f')$

$\vec{r_2} = (-g\sin v, g\cos v, 0)$

how see that $\vec{r_1}, \vec{r_2} = 0, \forall u, v.$

$\vec{r_1}, \vec{r_2} = -gg'\cos v\sin v + gg'\cos v\sin v - 0 = 0$

(i.e.,) the parameters are orthogonal the normal vector $\vec{\mathrm{N}}$ is given by

$\vec{N} = \frac{\vec{r_1}\times\vec{r_2}}{H}$, where $H = |\vec{r_1} \times \vec{r_2}|$

$\vec{r_1} = (g'\cos v, g'\sin v, f'g'\cos v, g'\sin v, f')$

$\vec{r}_2 = (-g\sin v_1, g\cos \mathrm{v}, 0{,}2\sin v\cos v, 0)$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ g'\cos v & g'\sin v & f' \\ -g\sin v & g\cos \mathrm{v} & 0 \end{vmatrix}$$

$= \vec{\imath}(0 - f'g\cos v) - \vec{\jmath}(0 + f'\,g\sin v) + \vec{k}(gg'\cos^2 v + gg'\sin^2 v)$

$\vec{r}_1 \times \vec{r}_2 = (-f'g\cos v, -f'g\sin^2 v, gg')$

$$N = \frac{\vec{r_1} \times \vec{r_2}}{H}$$

$$N = \frac{(-f'\,\mathrm{g}\cos\mathrm{v}, -f'g\sin v, gg_1)}{g(f^{12} + g^{12})^{1/2}}$$

It's after convenient to take $g(u) = u.$ for example the right circular cone of semi-vertical angle α is given by $g(u) = u, f(u) = u\cot\alpha$

The Anchor Ring:

This is obtained by rotating a circle of radius ' a ' about a line in its plane at a distance $b > a$ from the centre

Let the axis of rotation be the z-axis let M be a point ion the generating curve that lies in the $x$oz plane. If m=g(u), 0, f(u)

then $\mathrm{g}(u) = x -$ coordinateof m

$sf(u) = z$ - coordinate of m

$= a\sin u$

Let v denote the angle of rotation. then the position vector of the point u, v is given by,

$$\vec{\gamma} = (g(u)\cos v, g(u)\sin v, f(u))$$
$$= (b + a\cos u\cos v, (b + a\cos u)\sin v, a\sin u)$$

Here the domain of u, v is given by, $0 \leqslant u \leqslant 2\pi$, $0 \leq v \leq 2\pi$. Also the parameter curves $u =$ constant $v =$ constant are circles and

$$\vec{r_1} = \frac{d\vec{r}}{du} = (-a\sin u\cos v, -a\sin u\sin v, \mathrm{a}\cos \mathrm{u})$$

$$\vec{r_2} = \frac{d\vec{\gamma}}{dv} = (-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0)$$

$$\vec{r_1} \cdot \vec{r_2} = 0$$

Hence both the system of parametric curves is orthogonal.

The normal vector $\vec{N}$ is given by,

$$\vec{N} = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1} \times \vec{r_2}|}$$

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -a\sin u\cos v & -a\sin u\sin v & a\cos u \\ -(b + a\cos u)\sin v & (b + a\cos u)\cos v & 0 \end{vmatrix}$$

$$\vec{r_1} \times \vec{r_2} = (0 - a\cos u(\cos v)\,(b + a\cos u) - a\cos u\sin v(b + a\cos u) - a\sin u(b + a\cos u)$$

$$\vec{N} = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$

2.4.Helicoids:

A helicoid is a surface, generated by the Screw motion of a curve about a fixed line known as the axis.

The various position of a generating curve are obtained by first translating it through a distance λ, parallel to the axis and then rotating through an angle V about the axis, and then rotating through an angle v about the axis, where $\frac{\lambda}{v}$ has a constant value ' d '.

The distance travelled in one complete revolutions is $2\pi a$.

The constant $2\pi a$ is called the pitch of the helicoid. The pitch is positive or negative arc as the helicoid is right (or) left handed and the pitch is zero, for a given surface of revolution.

Equation the General Helicoids:

The section of the surface by the planes containing the taxis are congruent plane curves and the Surface is generated by the Screw motion of anyone of the curves

There is no loss of generality, if the generating curve is assumed to be plane. which is given by the equations of the form, $x = g(u), y = 0, z = f(u)$.

Let Γ be the point on the helicoid that corresponds to (u, v). where v is the angle of rotation.

Then we have $xp = OQ$.

$$= g(u)\sin(90° - v)$$
$$x_p = g(u)\cos v$$
$$y_p = OR$$
$$= g(u)\cos(90° - v)$$
$$y_p = g(u)\sin v$$

∴ The position vector of any point on the generate helicoid given by,

$\vec{r} = (g(u)\cos v, g(u)\sin v, f(u) + av)$

Now the parametric curves, V_E constant are the various le position of the pic

$u =$ constant are the circular helices.

Diff. with respect to r 'u

$\vec{r_1} = (g'(u)\cos v, g'(u)\sin v, f'(u))$

Diff. with respect to r

$\vec{r_2} = (-g(u)\sin u, g(u)\cos v, a)$

The parametric curves are orthogonal if, $\vec{r_1} \cdot \vec{r_2} = 0 \Rightarrow af'(u) = 0$.

$a = 0 \Rightarrow$ It is a surface revolution

$f'(u) = 0 \Rightarrow f(u) =$ constant

⇒ The helicoid is rigid helicoid.

Definition: Right Helicoid

This is helicoid, generated by a screw motion of a curve, straight line which meets the axis at right angle.

Taking the axis as the z-axis, the position vector of a points is, $\vec{\gamma} = (u\cos v, u\sin v, av)$

where u is the distance from the axis and v is the angle of rotation.

Here the generate being assumed to be to x-axis when $v = 0$.

$VM = x = u\sin(90 - v)$
$\quad\quad = u\cos v$
$OM = y = u\cos(90 - v)$
$\quad\quad = u\sin v$
$NP = \text{z} = \lambda = av$

Example 1:

A helicoid is generated by the Screw motion of a straight line skew to the axis. Find the curve coplanar with the axis, which generates the same helicoid.

Proof :

Let c be the Shortest-distance between the z-axis and the skew line.

Let α be the angle of relation between the axis and the straight line.

Then any point the skew line is,

$x = c$
$y = u\sin\alpha$
$z = u\cos\alpha$

Here, u is the distance of any point on the Skew line from to x-axis

To derive the equations of the helicoid

Let P denote the point on the helicoid obtained by the combination of rotation through an angle v about the z-axis and the translation with a distance a parallel to the axis.

∴ The position vector of any point p is given by,

$\vec{\gamma} = (c\cos v - u \sin\alpha \sin v, (\sin v + u\sin\alpha\cos v, u\cos\alpha + av)$

The required plane curve is the section of the Surface by the plane $y = 0$.

$c \sin v + u\sin\alpha\cos v = 0.$
$u\sin\alpha\cos v = -\sin v, c$

$u \sin\alpha = -c\tan v$

$x \ \cot\vec{r} = c \cos v + c \tan v \sin v$

$= c\cos v + \dfrac{c\sin^2 v}{\cos v} = \dfrac{c(\cos^2 v + \sin^2 v)}{\cos v}$

$= \text{c}\sec v\,,$

$y \text{ of } \vec{r} = 0$

$z \text{ of } \vec{r} = av + u\cos\alpha = av + u\cos\alpha \cdot \dfrac{\sin\alpha}{\sin\alpha}$

$= av + \cot\alpha + \cot\alpha\,(-c\ \tan v)$

$= av - c\cot\alpha\tan v$

2.5. Metric:

Obtain the expression for ds^2 where s is the arc length of a curve $u = u(t), v = v(t)$ on a surface $\vec{r} = \vec{r}(\vec{u}, \vec{v})$

Proof :

On a given surface $\vec{r} = \vec{r}(u, v)$

consider the curve given by,

$u = u(t), v = v(t)$

Then $\vec{r}$ is a functions of t along the curve and the arc length s is related to the parameter. ' t ' by the equations.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{d\vec{r}}{dt}\right)^2 = \left(\frac{\overrightarrow{dr}}{du}\cdot\frac{du}{dt} + \frac{d\vec{r}}{dv}\cdot\frac{dv}{dt}\right)^2$$

$$= \left(r_1\frac{du}{dt} + r_2\frac{du}{dt}\right)^2$$

$$= r_1^2\left(\frac{du}{dt}\right)^2 + 2r_1r_2\frac{du}{dt}\cdot\frac{dv}{dt} + r_2^2\left(\frac{dv}{dt}\right)^2$$

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\cdot\frac{du}{dt}\cdot\frac{du}{dt} + G\left(\frac{dv}{dt}\right)^2 \;\ldots\ldots\ldots\ldots(1)$$

where $E = \vec{r}_1^2, F = \vec{r}_1\vec{r}_2, G = \vec{r}_2^2$

Equations (1), becomes,

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$ .... (2)

we consider these quadratic differential For m as defined on the surface.

Geometrically (ds) can be interpreted as the infinite decimal distance from the point (u, v) to the point $(u + du, v + dv)$,

we see that, $(\overrightarrow{r_1} \times \overrightarrow{r_2}) = \vec{r}_1^2 \cdot \overrightarrow{r_2}2 - (\overrightarrow{r_1}\cdot\overrightarrow{r_2})^2$.

(ie) the co-effi Satisfy the eqns,

$H^2 = EG - F^2 > 0$

where $H = (\overrightarrow{r_1} \times \overrightarrow{r_2})$

(i.e.,) $H = \pm\sqrt{EG - F^2}$

Example 1:

For the paraboloid $x = u, y = v, z = u^2 - v^2$ find H.

Solution:

$\vec{r} = \left(u, v, (u^2 - v^2)\right)$

$\vec{r_1} = \dfrac{d\vec{r}}{du} = (1,0,2u)$

$\vec{r_2} = \dfrac{dr}{dv} = (0,1,-2v).$

$E = \vec{r}_1^2 = 1 + 0 + 4u^2 = 1 + 4u^2$

$G = \vec{r}_2^2 = 0 + 1 + 4v^2 = 1 + 4v^2$

$F = \vec{r_1} \cdot \vec{r_2} = 0 + 0 - 4uv$

$H = \sqrt{GE - F^2} = \sqrt{(1 + 4u^2)(1 + 4v^2) - 16u^2v^2}$

$H = \sqrt{1 + 4u^2 + 4v^2 + 16u^2v^2 - 16u^2v^2}$

Note:

when $F = 0$, (i.e.) $\vec{r_1} \cdot \vec{r_2} = 0$

This Show that the parametric curves are orthogonal.

Angle Between parametric curves:

The parametric directions are given $\vec{r}_1$ & $\vec{r_2}$ angle w. where $(0 < \omega < \pi)$ between them is given by,

$$\cos\omega = \frac{\vec{r_1} - \vec{r_2}}{|\vec{r_1}||\vec{r_2}|} = \frac{F}{(EG)^{1/2}}$$

$$\sin\omega = \frac{\vec{r_1} \times \vec{r_2}}{|\vec{r_1}||\vec{r_2}|} = \frac{H}{(EG)^{1/2}}$$

In general, the angle between the parametric directions varies from point to point.

Element of Area:

Consider the figure with four vertices $(u, v), (u + \delta u, v), (u + \delta u, v + \delta v), (u, v + \delta v) j$ joined by, the above figure is approximately a parallelogram with adjacent series $r_1 \delta u$ and $r_2 \delta v$

The area of the parallelogram

$$= |\vec{r_1}\delta u \times \vec{r_2}\delta v| = |\vec{r_1} \times \vec{r_2}|\delta u \delta v$$
$$= H\ \delta u \delta v$$

∴ The elementary area of the surface is given by $ds =$ H dvdu.

Example 2:

For the Anchor Ring in section 2.3:

$\vec{\gamma} = (b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u)$ os $0 \le u \le 2\pi, 0 \le v \le 2\pi$. Find the surface area of the anchor ring S

Solution:

We know that, $ds = H$ dudv. where $H = |\vec{r_1} \times \vec{r_2}|$.

$$\vec{r_1} = \frac{\partial \vec{r}}{\partial u} = (-a \sin u \cos v, -a \sin u \sin v, a\cos u)$$

$$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0)$$

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a \sin u \cos v & -a \sin u \sin v & a\cos u \\ -(b + a\cos u)\sin v & (b + a\cos u)\cos v & 0 \end{vmatrix}$$

$$= \vec{i}(-a(b + a \cos u) \cos u \cos v) - \vec{j}(a(b + a \cos u) \cos u \sin v$$

$$+\vec{k}(-a \sin u \,(b + a\cos u)\cos^2 v - a\sin u \,(b + a \cos u)\sin^2 v$$

$$\vec{r_1} \times \vec{r_2} = (-a\cos u\cos v(b + a\cos u), -a \sin v \cos u \,(b + a\cos u)\ )$$

$$-a(b + a\cos u)\sin u$$

$$|\vec{r_1} \times \vec{r}_2|^2 = a^2 \cos^2 u \cos^2 v \,(b + a\cos u)^2 + a^2\cos^2 u)\sin^2 v$$

$$(b + a\cos v)^2 + a^2\sin^2 u(b + a\cos u)^2$$

$$= a^2\cos^2 u(b + a\cos u)^2 + a^2\sin^2 u(b + a\cos u)^2$$

$$|\vec{r_1} \times \vec{r}_2|^2 = a^2(b + a \cos u)^2$$

$$H = |\vec{r_1} \times \vec{r_2}| = a(b + a \cos u)$$

∴ Surface area of the anchor ring

$$= \int_0^{2\pi} \int_0^{2\pi} H du dv$$

$$= \int_0^{2\pi} \int_0^{2\pi} (a(b + a) \cos u \, du \, dv).$$

$$= \int_0^{2\pi} \int_0^{2\pi} (ab + a^2\cos u)du \, dv \Rightarrow \int_0^{2\pi} a(b2\pi + a(0)dv$$

$$= \int_0^{2\pi} ab2\pi dv = ab \cdot 2\pi \cdot 2\pi = 4\,\pi^2\text{ab}$$

Metric is invariant under parametric Transformation:

Let $u' = \phi(u, v)$ and

$v' = \psi(u, v)$ be a prove that

$$\vec{r}_1' = \frac{\partial \vec{r}}{\partial u'} = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial u}{\partial u'} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial u'}$$

$$= \vec{r}_1 \frac{\partial u}{\partial u'} + \vec{r_2}\frac{\partial v}{\partial u'}$$

$$\overrightarrow{r_2}' = \frac{\partial\vec{r}}{\partial v'} = \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial u}{\partial v'} + \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial v}{\partial v'}$$
$$= \vec{r}_1\frac{\partial u}{\partial v'} + \overrightarrow{r_2}\frac{\partial v}{\partial u'}$$

In terms of the parameter $u' and\ v'$

$$E'(du')^2 + 2F'du'dv' + G'(dv')^2$$

$$=\vec{r}_1'^2(du')^2 + 2(\vec{r}_1', \overrightarrow{r_2})du'\cdot dv' + r_2'^2(dv')^2$$
$$= \left(\vec{r}_1'du' + \overrightarrow{r_2}'dv'\right)^2$$
$$= (\vec{r}_1'du' + \overrightarrow{r_2}'dv')$$
$$= \left\{\left(\overrightarrow{r_1}\frac{\partial u}{\partial u'} + \overrightarrow{r_2}\frac{\partial v}{\partial u'}\right)du' + \left(\overrightarrow{r_1}\frac{\partial u}{\partial v'} + \overrightarrow{r_2}\frac{\partial v}{\partial v'}\right)dv'\right\}^2$$

$$= \left\{\vec{r}_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + \vec{r}_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right\}^2$$
$$= \{\overrightarrow{r_1}du + \overrightarrow{r_2}dv\}^2$$
$$= \vec{r}_1^2(du)^2 + 2(\overrightarrow{r_1}\cdot\overrightarrow{r_2})dudv + \vec{r}_2^2(dv)^2$$
$$= E(du)^2 + 2Fdudv + G(dv)^2.$$

∴ The metric is invariant under the parameter transformation but the co-efficient E, F, G are not invariant.

2.6.Direction Coefficients:

At a point ' p ' of a surface, there are three independent vectors. $\vec{N}, \vec{r}_1$ and $\overrightarrow{r_2}$. Every vector $\vec{a}$ at f can therefore be expressed as,

$$\vec{a} = a_n\vec{N} + \lambda\overrightarrow{r_1} + \mu\overrightarrow{r_2}$$

[line of intersection of the plane containing $\vec{N}$ and $\vec{a}$ with the tangent plane at P]

Angles in the tangent plane:

Find the angle between two directions :

Angle in the tangent plane will be measured in the sense of relation. which carries the direction of $\overrightarrow{r_1}$ to the direction of $\overrightarrow{r_2}$ through an angle between 0 and π.

This is also the positive sense of relation about $\vec{N}$ [If (l, m) and (l', m') are the direction co-eff of two directions, at the same point the correspond unit vectors are

$$\vec{a} = l\overrightarrow{r_1} + m\overrightarrow{r_2}$$
$$\vec{b} = l'\overrightarrow{r_1} + \mathrm{m}'\overrightarrow{r_2}$$

∴ The angle between the direction is given by,

$\cos\theta = \vec{a}\cdot\vec{b} \ \times \vec{N}\cdot\sin\theta = \vec{a}\times\vec{b}$

we have,

$$\vec{a}\cdot\vec{b} = (l\vec{r_1} + m\vec{r_2})\cdot(l'\vec{r_1} + m'\vec{r_2})$$
$$= ll'\vec{r}_1^2 + \left(lm' + l'm\right)\overrightarrow{r_1 r_2} + mm'\vec{r_2}$$
$$\cos\theta = Ell' + F(lm' + l'm) + Gm' \quad \dots\dots\dots\dots(3)$$
$$\vec{a}\times\vec{b} = (l\vec{r_1} + m\vec{r_2})\times(l'\vec{r_1} + m'\vec{r_2})$$
$$= lm'(\vec{r_1}\times\vec{r_2}) + l'm(\vec{r_2}\times\vec{r_1})$$
$$= (\vec{r_1}\times\vec{r_2})(lm' - l'm)$$
$$|\vec{a}\times\vec{b}| = |\vec{r_1}\times\vec{r_2}|(lm' - l'm)$$
$$\Rightarrow |N\cdot\sin\theta| = H(lm' - l'm)$$
$$\therefore \sin\theta = H(lm' - l'm) \quad \dots\dots\dots\dots.(4)$$

Example 1:

Find the coefficient of the direction which makes an angle $\frac{\pi}{2}$ with the direction whose co-efficients are (l, m). [Interms of given direction (λ, m)].

Solution:

Let (l', m') be the required direction co-efficient in the direction. which makes an angle $\pi/2$ with the given direction.

$$\cos\theta = Ell' + F(lm' + l'm) + G(m')$$

$$\sin\theta = H'(lm' - l'm)'.$$

Then from (3) & (4)

$0 = Ell' + F(lm'$ +(n) $+Gmm'$ ……………. (i)

$I = H(\ell m' - l'm) \quad \dots\dots\dots\dots(ii)$

From (i)

$$l'(El + Fm) + m'(Fl + Gm) = 0$$
$$\Rightarrow l'(El + Fm) = -m'(Fl + Gm)$$
$$\Rightarrow \frac{l'}{(Fl + Gm)} - \frac{m'}{El + Fm} = \alpha$$
$$\Rightarrow l' = -\alpha(Fl + Gm)$$
$$m' = \alpha(El + Fm)$$

Sub (ii)

$1 = H(l\alpha(El + Fm) + m\alpha(Fl + Gm))$

$1 = \alpha H(El^2 + 2Flm + Gm^2)$
$1 = \alpha H(1)$
$\alpha = \frac{1}{H}$

$l' = -\frac{1}{H}(Fl + Gm)$

$\left[m' = \frac{1}{H}(El + Fm)(l', m')\right.$ are indeed directed co-efficient.

$$El'^2 + 2Fl'm' + Gm'^2 = \frac{E}{H^2}(Fl + Gm)^2 - \frac{2F}{H^2}(Fl + Gm)(El + Fm) + \frac{G}{H^2}(El + Fm)^2$$

$$= \frac{1}{H^2}[EF^2l^2 + 2EFGlm + EGG^2m^2 - 2EF^2l^2 - 2F^3l - 2EFGlm$$

$$-2F^2Gm^2 + E^2Gl^2 + 2EFGlm + F^2GM$$

$$= \frac{1}{H^2}[El^2(EG - F^2) + 2Flm \in EG - F^2) + Gm^2(EG - F^2)$$

$$= \frac{EG - F^2}{EG - F^2}[El^2 + 2Flm + Gm^2] \quad H = \pm\sqrt{EG - F^2}$$

$= (\because H^2 = EG - F^2)$
$= 1$

$[\therefore (\ell', m')$ direction coefficient $\Rightarrow El'^2 + 2Fl'm' + Gm'^2 = 1]$.

Exercises:

Find the identity satisfied by direction coefficient in relation to the co-efficient of the metric ds ^{2}. Find the angle between two directions obtain (l, m) in terms of given directions ratio (λ, μ)

(or)

On a surface $\vec{r} = \vec{r}_{(u, 0)}$. Let u over line u=u(t), and v = v(t) respectively a curve obtain an expression for the angle between them and also find the elemental area in terms of the co-efficient of the metric ds^2. use this to complete the area of whole anchor ring

g(u) = b + a cos u, f(u) = a sin u

2.7. Families of Curves:

Let $\phi(u, v)$ be a single valued function of u,v possessing continuous partial derivative ϕ_1 , ϕ_2 which do not vanish. Then the implicit equation $\phi(u, v) = c$ where c is a real parameter gives a family of curves on the surface $\vec{r} = \vec{r}(u, v)$

Properties:

i) Through every point (u, v) on the surface there passes one and only member of the family.

ii) Let $\phi(u_0, v_0) = c_1$ where (u_0, v_0) is any point on the surface. Then $\phi(u_0, v_0) = c_1$ is a member of the family passing through (u_0, v_0). Hence through every point on the surface, there passes one and only one member of the family.

iii) The direction ratios of the tangent to the curve of the family at (u, v) is $(-\phi_2 , \phi_1)$.

Theorem 1:

The curve of the family ϕ(u, v)=constant are the solution of the differential equation $\phi_1$du+ ϕ_2 dv=0 ......(1) and conversely a first order differential equation of the form P(u, v)du + Q(u, v)dv=0 .....(2) where P and q are differential functions which do not vanish simultaneously define a family of curves.

Proof:

Since $\phi_1 = \frac{\partial\phi}{\partial u}$ and $\phi_2 = \frac{\partial\phi}{\partial v}$, we get from (1), $\frac{\partial\phi}{\partial u}\, du + \frac{\partial\phi}{\partial v}\, dv = 0$ =0 giving $d\phi$ =0

Hence we conclude that ϕ (u, v)=c. Thus as the constant c varies, the curves of the family are the different solutions of the differential equation.

Conversely let us consider the equation (2). Unless the equation is exact, it is not in general possible to find a single function ϕ (u, v) such that ϕ_1=P and ϕ_2=Q.

However we can find integrating factor λ(u ,v) such that $\phi_1 = P\lambda$ and $\phi_2 = Q\lambda$. .

Rewriting the equation (2) in the form $\lambda P\, du + \lambda Q\, dv = 0$, we get $\phi_1$du+ ϕ_2 dv=0, so that the solution of the equation is $\phi(u, v) = c$.

Further from (2), $\frac{du}{dv} = -\frac{Q}{P}$ so that the direction ratios of the tangent to the curves of the family at the point P is (-Q,P).

Theorem 2:

For a variable direction at P, $\left|\frac{d\phi}{ds}\right|$ is maximum in a direction orthogonal to the curve ϕ(u, v)=constant through P and the angle between $(-\phi_2 , \phi_1)$ and the orthogonal direction in which ϕ is increasing is $\frac{\pi}{2}$.

Proof:

Let P (u, v) be any point on the surface. We shall show that ϕ increases most rapidly at P in a direction orthogonal to the curve of the family passing through P. For this, we prove that $\frac{d\phi}{ds}$ has the greatest value in such a direction.

Let (l, m) be any direction through P on the surface. Let μ be the magnitude of the vector

$\phi = (-\phi_2 , \phi_1)$. Let θ be the angle between (l, m) and the vector ϕ.

Let us take $a = \text{l}r_1 + \text{m}r_2$, $b = -\phi_2 r_1 + \phi_1 r_2$

We shall find $a \times b$ expressing sinθ in terms of H and μ =|b| where

From the definition $|a| = 1$.

We have $|a \times b| = \mu \sin\theta$ ....(1)

and $a \times b = (l\phi_1 + m\phi_2)(r_1 \times r_2)$ so that

$|a \times b| = H(l\phi_1 + m\phi_2)$ ....(2)

Equating (1) and (2), we obtain

$\mu \sin\theta = H(l\phi_1 + m\phi_2)$.............. (3)

Since (l, m) are the direction coefficient of any direction through P, we have

l= $\frac{du}{ds}$, m=$\frac{dv}{ds}$ ....(4)

Using (4) in (3) and simplifying, we get $\mu \sin\theta = H\frac{d\phi}{ds}$

Now μ and H are always positive and do not depend on (l, m).

Hence $\frac{d\phi}{ds}$ has maximum value $\frac{\mu}{H}$ when sin θ has maximum value in which case $\theta = \frac{\pi}{2}$.

In a similar manner, $\frac{d\phi}{ds}$ has minimum value -$\frac{\mu}{H}$, when $\theta = -\frac{\pi}{2}$. Since H >0 and $\mu > 0$, the orthogonal direction for which $\frac{d\phi}{ds}$ >0 is such that $\theta = \frac{\pi}{2}$.

Hence $\left|\frac{d\phi}{ds}\right|$ has maximum in a direction orthogonal to ϕ(u, v)=constant.

Orthogonal Trajectories:

For a given family of curves, the always exists a second family trajectories such that at every point two curves one from each family are orthogonal.

Problems:

(i)Prove that every family of curves on a Surface possess orthogonal trajectories.

(ii) The parameters on a surfaces can always be chosen such that the curves of a given family and their orthogonal trajectories between parametric curves.

Proof:

(i) Let the given family is defined by $P(u, v)du + Q(u, v)dv = 0$ ....(1)

where P and Q are functions of u and v class 1 & P & Q do not vanish together.

From (1), $\frac{du}{dV} = \frac{-Q}{P}$

(i.e.) $(-Q, P)$ are the directions of the tangent at (u, v) of a member of the family is given by, $\phi(u, v)$ = constant.

Such that $\phi_1 = \lambda p$ and $\phi_2 = \lambda Q$

Let (du, dv) be the differential in an orthogonal direction to the tangent at (u, v) for the curve.

$\phi(u, v) =$ constant

$\text{Cos}\theta = Ell' + F(lm' + l'm) + Gmm'.$

$\therefore \cos 90° = 0 = E(-Q)du + F(-Qdv + Pdv) + GPdv$

$0 = du(FP - EQ) + dv(\text{GP} - \text{F}Q)$ ………… (2)

$\therefore P$ and Q are functions of class 1.

Further,

$(FP - EQ)$ and $(GP - FQ)$ do not vanish together (2) is integral.

$\Rightarrow \exists$ functions $\mu(u, v) \neq 0$ and $\psi(u, v) \neq 0$.

Such that $\mu(FP - FQ) = \psi_1$

$\mu(GP - FQ) = \Psi_2$

$\Rightarrow \Psi(u, v) =$ constant is the equs of the orthogonal frajection of the given family of curve $\phi(u, v) =$ constant .

(ii) We have $\frac{\partial(\phi,\psi)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} \phi_1 & \phi_2 \\ \psi_1 & \psi_2 \end{vmatrix}$

$= \begin{vmatrix} \lambda P & \lambda Q \\ \mu(FP - EQ) & \mu(GP - F(Q) \end{vmatrix}$

$= \lambda\mu(GP^2 - FPQ - FPQ + EQ^2].$

$= \lambda\mu[GP^2 - 2FPQ + EQ^2] \neq 0$

The quadratic $GP^2 - 2FPQ + EQ^2$ is positive, when $\lambda \neq 0, \mu \neq 0$, and as, P, Q do not vanish together.

$\left. \begin{array}{l} u' = \phi(u, v) \\ v' = \psi(u, v) \end{array} \right\}$ ……………(3)

(i.e.) $\phi(u, v) =$ constant, $\Rightarrow u' =$ constant

$\psi(u, v) =$ constant, $\Rightarrow v' =$ constant.

Thus by the given family,

$\phi(u, v) =$ constant & its orthogonal trajectory curves given by, $u' =$ constant, $v' =$ constant.

Example 1:

On the parabolic $x^2 - y^2 = z$,Find the orthogonal trajectories of the section of the plane z =constant

Proof:

Let $x = u, y = v$

we get, $x^2 - y^2 = z$

$u^2 - v^2 = z$

$\therefore$ The surface is given by, $\vec{r} = (u, v, u^2 - v^2)$

Now, $z =$ constant $\Rightarrow u^2 - v^2 =$ constant $= c^2$ (say).

The differential equations is $2udu - 2vdv = 0$

$$udu = vdv$$
$$\frac{du}{dv} = \frac{v}{u}$$

$\therefore$ The direction of the tangent to the curve belonging to the family at (u, v) is (v, u) . If (du, dv) are the differentiable in an orthogonal direction to the direction of $f(u, v)$ then we have, $l(l, m) = (v, u)$

$\text{Cos}90° = 0 = Ell' + F(l' + l' + m) + Gmm'$

$0 = Evdu + F(vdv + udu) + \text{G udv}$ ... . (1)

$\therefore \ \vec{r_1} = \frac{d\vec{r}}{du} = (1,0,2u) \ \because \vec{r} = (u, v, u^2 - v^2)$

$\vec{\gamma}_2 = \frac{dr}{dv} = (0,1,-2v)$

$E = \vec{r}_r^2 = 1 + 4u^2$

$G = \vec{r}_2^2 = 1 + 4v^2$

$F = \vec{r_1} \cdot \vec{r_2} = 0 + 0 - 4uv$

From (1) $\Rightarrow 0 = (1 + 4u^2)vdu + (-4uv)(udu + vdv)t + (1 + 4v^2)udv$

$$0 = vdu + 4u^2vdu - 4u^2\text{v}du - 4uv^2dv + udv + 4v^2udv$$
$$0 = vdu + udv$$
$$= d(u, v)$$

$\therefore uv =$ constant

Which is the required orthogonal trajectories if the family of curves $u^2 - v^2 =$ constant

Example 2:

A helicoid is generated by the skew motion of a straight line. which meets the axis at an angle α. Find the orthogonal trajectories of the generators. Find the de also metric of the surface referred to the generation and their orthogonal trajectories as parametric curves.

Proof:

(i)Let z - axis be the axis of the helicoid.

Let the generating line OA, makes an angle α with Oz.

Any point p on OA has co-ordinates ($u \sin\alpha, 0, u\cos\alpha$) where op=u

Let the line OA be translated through a distance a parallel to Oz and then be rotates through an angle 'av' about the z-axis.

Let Q be the position of p under the transformation.

∴ we have,

$$ZQ = zp + av$$
$$= u\cos\alpha + av$$
$$xQ = OR\sin(90° - v)$$
$$xQ = u\sin(\alpha)\cos v$$
$$yQ = OR\cos(90 - v)$$
$$yQ = u \sin\alpha \sin v$$

(i.e.) Q has coordinates,

$(u \sin\alpha \cos v, u \sin\alpha \sin v, u \cos\alpha + \mathrm{av})$

∴ The position vector of a point Q is

$\vec{r} = (u \sin\alpha \cos v, u \sin\alpha \sin v, u\cos\alpha + av)$

Now the generator of the helicoid is,

$\mathrm{v} =$ constant

$dv = 0$

Now the direction of the tangent to the curve belong to the family at (u, v) is given by, $(1,0)$

If (du, dv) are differentials, then by the formula,

$$\cos 90° = 0 = Ell' + F(lm' + \ell'm) + Gmm'$$
$$0 = Edu + Fdv \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now, $\vec{r}_1 = \frac{d\vec{r}}{du}$

$$\vec{r_1} = (\sin\alpha \cos v, \sin\alpha \sin v, \cos\alpha)$$
$$\vec{r_2} = (-u \sin\alpha\ sin\ v, u\sin\alpha\cos v, a).$$
$$E = \vec{r}_1^{\,2} = \sin^2\alpha\cos^2 v + \sin^2\alpha \sin^2 v + \cos^2\alpha = 1$$
$$F = \vec{r_1}\vec{r_2} = -u \sin^2\alpha \cos v\sin v + u \sin^2\alpha \sin v \cos v + a\cos\alpha$$
$$F = \vec{r_1} \cdot \vec{r_2} = a \cos\alpha.$$

From (1) $\Rightarrow 0 = 1du + a\cos\alpha dv$ on integration.

$C = u + a\cos\alpha v$

This is the equations of the orthogonal trajectories of the generated of the helicoid.

(ii) If the generator $v=$ constant

$u+a\cos\alpha v=$ constant are taken as parametric u' & v',

$u'=u+a\cos\alpha v$ and $v'=v$

from this equations, we have,

$u=u'-a\cos\alpha v'$ and $v=v'$

In this ways, generators and the orthogonal trajectories become parametric curves $u'=v'=$ constant.

The metric referred to this new parametric

$$ds^2=E'(du')^2+2F'du'dv'+G'(dv')^2 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

To calculate $E'F'G'$

$$\vec{r}_1'=\frac{\partial\vec{r}}{\partial u'}=\frac{\overrightarrow{\partial r}}{\partial u}\cdot\frac{\partial u}{\partial u'}+\frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial v}{\partial v'}=\vec{r}_1(1)+\overrightarrow{r_2}(0)=\overrightarrow{r_1}$$

$$\vec{r}_2'=\frac{\partial\vec{r}}{\partial v'}=\frac{\partial r}{\partial u}\cdot\frac{\partial u}{\partial v'}=\vec{r}_1(-a\cos\theta)+\vec{r}_2(1).$$

$$E'=\overrightarrow{r_1}'^2=\overrightarrow{r_1}^2=1, \qquad E'=1$$

$$F'=\overrightarrow{r_1}'\cdot\overrightarrow{r_2}^2=\overrightarrow{r_2}\cdot\overrightarrow{r_2}-a\cos\alpha\vec{r}_1^2$$

$$=a\cos\alpha-a\cos\alpha\cdot 1=0.$$

$$G=\vec{r}_2{}^2=a(\overrightarrow{r_2}-a\cos\alpha\overrightarrow{r_1})^2$$

$$=\vec{r}_2^2-2a\overrightarrow{r_2}^2\cos\alpha\overrightarrow{r_1}+a^2\cos^2\alpha\vec{r}_1^2$$

$$=(u^2\sin\alpha+a^2)-2a\cos\alpha(a\cos\alpha)+a^2\cos^2\alpha,1$$

$$=u^2\sin^2\alpha+a^2-a^2\cos^2\alpha$$

$$=a^2\sin^2\alpha+a^2\sin^2\alpha$$

$$G'=(u^2+a^2)\sin^2\alpha$$

$$G'=((u'-a\cos\alpha v')^2+a^2)\sin^2\alpha$$

From (2) $ds^2=1\cdot du^2+0\cdot du'dv'+\big((u'-a\cos\alpha v')+u^2\sin^2\alpha dv^2$

$ds^2=\mathrm{du}^2+\big((u^1-a\cos \mathrm{v}\, v^1+a^2)\sin^2\alpha dv^2$

Double Family of curves:

If P,Q,R are, continuous functions of u & v.

Which do not vanish to getter, the quadratic differential. equations

$P(du)^2+2Qdudv+R(dv)^2=0 \quad \ldots\ldots\ldots(*)$

represents two family of curves on the surfaces provided $Q^2-PR>0$.

(ie) in (*) discriminate $=4Q^2-4PR>0$

$=Q^2-PR>0$

For example, $du^2-5dudv+6dv^2=0.\Rightarrow(du-3dv)(du-2du)=0.$

$\Rightarrow du - 3dv = 0 \;\&\; du - 2dv = 0$

Thus we get two family of curves

To find the condition that the quadratic equations

$P(du)^2 + 2Qdudv + R(dv)^2 = 0$ …..(1) represents orthogonal family of curves.

Proof:

From equation(1).

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0$$

If (λ, μ) and (λ', μ') are the directions of the tangent of the two family of curves, then the roots of equations (2) are $\frac{\lambda}{\mu}$ & $\frac{\lambda'}{\mu'}$.

Hence sum of the roots $= \frac{\lambda}{\mu} + \frac{\lambda'}{\mu'} = \frac{-2Q}{P}\}$ ………(3)

product of the roots $= \frac{\lambda}{\mu} \cdot \frac{\lambda'}{\mu'} = \frac{R}{P}$

We know that the condition for orthogonality,

$$\cos 90° = 0 = E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu'$$

$$0 = E\left(\frac{\lambda}{\mu} \cdot \frac{\lambda'}{\mu'}\right) + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G$$

$$0 = E\frac{R}{P} + F\left(\frac{-2Q}{P}\right) + G.$$

$$0 = ER - 2QF + GP$$

which is the required condition.

Note:

If $R = P = 0$ then from (1), $du, dv = 0$ then the condition for orthogonality is $\mathrm{QF} = 0$.

$F = 0$.

Example 3:

If θ is the angle at the point (u, v) between the two directions given by the equations.

$P(du)^2 + 2Qdudv + R(dv)^2 = 0$ ………….. (1)

then prove that $\tan\theta = \frac{2H(Q^2 - PR)^{1/2}}{ER - 2QF + GP}$

Proof:

Let the roots of equations (1) are $\frac{\lambda}{\mu}$ & $\frac{\lambda'}{\mu'}$.

Hence the sum of the roots $= \frac{\lambda}{\mu} + \frac{\lambda'}{\mu'} = \frac{-2Q}{P}$, product of the roofs $= \frac{\lambda}{\mu} \cdot \frac{\lambda'}{\mu'} = \frac{R}{P}$ ……(2)

$$= \frac{H(\lambda\mu' - \lambda'\mu)}{E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G(\mu\mu')}$$

$$= \frac{H\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right)}{E\frac{\lambda\lambda'}{\mu\mu'} + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G}$$

$$\tan\theta = \frac{H\left[\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right)^2 - 4\frac{\lambda\lambda'}{\mu\mu'}\right]^2}{\frac{ER}{P} + F\left(\frac{-2Q}{P}\right) + G}$$

$$= \frac{\frac{2H}{P}\left[(Q^2 - PR)^{1/2}\right]}{\frac{ER - QFQ + GP}{P}}$$

$$\tan\theta = \frac{2H(Q^2 - PR)^{1/2}}{ER - 2FQ + GP} \Rightarrow ER - 2FR - GP = 0$$

2.8.Isometric Correspondence:

Two surface s and s' are said to be isometric. If there exists a correspondence, $u' = \phi(u, v)$ and $v' = \psi(u, v)$ between their parameters, where ϕ & ψ are single valued functions and $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$.

Such that the metric of S transform into the metric the correspondence itself is an isometry.

Note:

If the two Surface are isometric then the length of the arcs of the corresponding on the surfaces are equal.

Theorem 1:

To each direction of the tangent to a curve C at P in S, there corresponds a direction of the tangent C' at P' in S' and vice-versa.

Proof:

Let C be a curve of a class ≥1 passing through P and lying on S. Let it be parametrically represented by u=u(t) and v=v(t). If is the portion corresponding to S under the relation (1) in the preceding paragraph, then C on S will be mapped onto C' on S' passing through $P'$with the parametric equations

$u' = \phi\{u(t), v(t)\}$,

$v' = \psi\{u(t), v(t)\}$,

The direction ratios of the tangent at P to C are $(\dot{u}, \dot{v})$where $\dot{u} = \frac{du}{dt}, \dot{v} = \frac{dv}{dt}$

Now the direction ratios of the tangents at $P'$to $C'$are $(\dot{u}', \dot{v}')$where

$$\dot{u}' = \frac{du'}{dt} = \frac{\partial\phi}{\partial u}\dot{u} + \frac{\partial\phi}{\partial v}\ \dot{v}$$

$$\dot{v}' = \frac{dv'}{dt} = \frac{\partial\psi}{\partial u}\dot{u} + \frac{\partial\psi}{\partial v}\ \dot{v}$$

Solving the above equation for $\dot{u}$ and $\dot{v}$, we get ,

$\dot{u} = \frac{1}{J}\left(\dot{u}'\frac{\partial\psi}{\partial v} - \dot{v}'\frac{\partial\phi}{\partial v}\right),\ \dot{v} = \frac{1}{J}\left(\dot{v}'\frac{\partial\phi}{\partial u} - \dot{v}'\frac{\partial\psi}{\partial u}\right)$where J $\neq 0$

which shows that a given direction to a curve C' at P'corresponds to a definite direction at P to C and vice-versa.

Example 1:

Find the Surfaces of the revolution of the right helicoid of pitch $2\pi a$.

Proof:

Let the surface of revolution be given by,

$$\vec{r} = \left(g(u)\cos v\,, g(u)\sin v\,, f(u)\right)$$

$$\overrightarrow{r_1} = \frac{\partial\vec{r}}{\partial u} = (g'(u)\cos v\,, g'(u)\sin v\,, f'(u).$$

$$\overrightarrow{r_2} = \frac{\partial\vec{r}}{\partial v} = (-g(u)\sin v\,, g(u)\cos v\,, 0).$$

$$\therefore E = \vec{r}_1^{\,2} = (g'(u)^2\cos^2 v + g'u^2\sin^2 v + f'(u))^2$$
$$= (g'(u))^2 + (f'(u))^2$$
$$G = \vec{r}_2^{\,2} = (g(u))^2\sin^2 v + (g(u))^2\cos^2 v.$$
$$G = (g(u))^2$$
$$F = \overrightarrow{r_1}\cdot\overrightarrow{r_2} = -g(u)g'(u)\sin v\cos v + g(u)g'(u)\sin v\cos v$$
$$\Rightarrow F = 0$$

The metric of the surface of revolution is given by,

$$Edu^2 + 2Fdudv + Gdv^2 = 0$$
$$[(g'(u))^2 + (f'(u))^2du^2 + (g(u)^2dv^2 = 0 \dots\dots\dots.(1)$$

The equations of the right helicoid is given by,

$$\vec{r}' = (u'\cos v', u'\sin v', av')$$

$$\overrightarrow{r_1}' = \frac{\partial\vec{r}}{\partial u'} = (\cos v'\sin v', 0)$$

$$\vec{r}_2' = \frac{\partial\vec{r}}{\partial v'} = (-u'\sin v', u\cos v', a)$$

$$E' = \vec{r}_1'^2 = (\cos' v)^2 + (\sin v')^2 = 1$$
$$G' = \vec{r}_2'^2 = u'^2 + a^2.$$
$$G' = u'^2 + a^2.$$
$$F' = \vec{r}_1'\cdot\overrightarrow{r_2} = -u'\sin v'\cos v' + u'\sin v'\cos v' + 0$$
$$F' = 0$$

The metric of the right helicoids is

$$E'du'^2 + 2F'du'dv' + G'dv'^2 = 0$$
$$1(du')^2 + ((u')^2 + a^2)dv'^2 = 0 \quad \ldots\ldots\ldots\ldots (2)$$

To find a transformation from $(u, v) \to (u', v')$.

Let $v = v' \to dv' = dv$

$$u' = \phi(u) \longrightarrow du' = \frac{\partial \phi}{\partial u} \cdot du$$
$$du' = \phi' \cdot du$$

Sub in (2).

$$(\phi')^2 \mathrm{du}^2 + ((\phi(u))^2 + a^2)dv^2 = 0 \quad \ldots\ldots\ldots\ldots (3)$$

Equation (1) & (2) are identical.

$$(g'(\mathrm{u}))^2 + (f'^{(\mathrm{u}))^2} = (\phi')^2 \quad \ldots\ldots\ldots\ldots (4)$$
$$g(u))^2 = (\phi(u))^2 + a^2 \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$\therefore (g(u))^2 = a^2 \sinh^2 u + a^2$

$= a^2(\sinh^2 u + 1)$

$(g(u))^2 = a^2 \cosh h^2 u$

$g(u) = a\cosh u$

Now, $\phi'(u) = a\cosh u = \frac{\partial \phi}{\partial u'}$.

From (4), $a^2 \sinh^2 u + (f'(u))^2 = a^2 \cosh h^2 u$

$$(f'(u))^2 = a^2(\cosh^2 u - \sinh^2 u)$$
$$= a^2$$
$$f'(u) = a$$

Integrating, $f(u) = \mathrm{au}$

The rigid helicoid is isometric with the surface obtained by revolving the curves.

x = a$\cosh u$, y=0, z= au about z-axis.

2.9. Intrinsic properties:

Statement of Existence Theorem:

If E,F,G are any given single valued functions with E>0 and $EG - F^2 > 0$ in the domain D. Then every point of D has a neighbourhood D' in which $Edu^2 + 2\,F\,du\,dv + G\,dv^2$ is the metric of the surface referred to u, v as parameters.

Properties:

(i)Any two isometric surface have the same metric S, when the corresponding points are arranged with the

(i.e.) The family of surfaces, having a given metric is the class of surfaces isometric to one another.

ii)A surface which is reducible from the metric using the vector equations $\vec{r} = \vec{r}(u, v)$ applies to the whole class of isometric surfaces of the same kind with the same properties.

Example 1:

Find the surface of revolution which is isometric with a region of the right helicoid.

Proof:

Two surfaces S, S' are said to be isometric (or) applicable if there is a correspondence between the points of s and S'. Such that corresponding arcs of curves have the same length.

The correspondence is called an isometry.

We know that a surface of revolution is given by

$r = (g(u)\cos v, g(u) sinv, f(u)) \dots \dots ..(1)$

For some function f and g and its metric is $(g_1^2 + f_1^2)du^2 + g^2 dv^2$

Where $f_1 = \frac{du}{dt}$.

The right helicoid of pitch $2\pi a$ is given by

$r = (u' \cos v', u' \sin v', \mathrm{a}\, v')$ and its metric is

$du'^2 + (u'^2 + a^2)dv'^2$

We have to find the transformation $(u, v) \longrightarrow (u', v')$ which makes two metrics identical.

Taking $v' = v, u' = \phi(u)$ then $du' = \phi_1 du$ and the metrics are identical.

If $g^2 = \phi^2 + a^2,\ g_1^2 + f_1^2 = \phi_1^2$

These are two equations for three functions namely, f, g, and ϕ.

If $\phi\ is$ eliminated there remains a differentiation equation for f as a function of g.

(or) Simply put $\phi(u) = a \sin hu\ \ and\ g(u) = a \cos hu$ to satisfy equation (1),

$f_1^2 = a^2$ we can take f(u)=au

Hence, the right helicoid is isometric with the surface obtained by revolving the curve.

$x = a \cos hu$

$y = 0$

$z = au$ about z-axis the generating curve is,

$x = a \cos h(\frac{z}{a})$ with the parameter a and directrix to z-axis and the surface of revolution is a catenoid,

$u' = a \sin hu$

$v' = v$

Shows that the generators v'=constant on the helicoid

And v'= constant on the catenoid

And the helices u'=constant correspond to the parallel u=constant.

On the helicoid u' and v' can take all values but on the catenoid $0 \leq v \leq 2\pi$

The correspondence is therefore an isometry only for that region of the helicoid for which

$0 \leq v' \leq 2\pi$

Without the limitation to one period of the helicoid the correspondence would be locally isometric.

UNIT-III:

Geodesics: Geodesics – Canonical geodesic equations – Normal property of geodesics- Existence Theorems – Geodesic parallels – Geodesics curvature- Gauss- Bonnet Theorem – Gaussian curvature- surface of constant curvature.

Chapter 3: Sections 3.1 - 3.8.

3.1. Geodesics:

On any surface there are special intrinsic curves, called geodesics, which are analogous to straight lines in Euclidean space because they are curves of shortest distance. The problem is, given any two points A and B on the surface, to find, out of all the arcs joining A and B, those which give the least arc length. This problem, treated properly, is difficult and beyond the scope of this book. For example, it is by no means clear that a solution exists, for although the lengths of the various arcs AB certainly have a non-zero greatest lower bound, it does not follow that there is an arc of this length. However, the problem does lead to a definite answer in the form of differential equations for the functions $u = u(t), v = v(t)$ defining the curve. Every curve given by these equations is called a geodesic, whether it is a curve of shortest distance or not, and geodesics may be regarded as curves of stationary rather than strictly shortest distance on the surface.

We shall now derive the geodesic differential equations mentioned above by formulating a more restricted problem.

Let A, B be any two points, and consider the arcs which join A and B and are given by equations of the form $u = u(t), v = v(t)$ .where $u(t)$ and $v(t)$ are of class 2 . Without loss of generality it can be assumed that for every arc $\alpha, t = 0$ at A and $t = 1$ at B, so that α is given by $0 \leqslant t \leqslant 1$. Then the length of α is

$$s(\alpha) = \int_0^1 \left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)^{\frac{1}{2}} dt, \ldots\ldots\ldots\ldots\ldots (1)$$

where $u(t)$ and $v(t)$ are substituted for u and v in $E, F,$ and G. Suppose now that an are α' is obtained by deforming α slightly, keeping its end points A and B fixed. Then α' is given by equations of the form

$$u = u'(t) = u(t) + \epsilon\lambda(t), v = v'(t) = v(t) + \epsilon\mu(t)$$

where ϵ is small, and λ and μ are arbitrary functions of t of class 2 in $0 \leqslant t \leqslant 1$ and satisfying $\lambda = \mu = 0$ at $t = 0$ and $t = 1$. The length of α' is $s(\alpha')$ given by (1) with u', v' in place of u, v. The variation in $s(\alpha)$ is $s(\alpha') - s(\alpha)$ and is in general of order ϵ. If, however, α is such that the variation in $s(x)$ is at most of order ϵ^2 for all small variations in α (i.e. for all $\lambda(t)$ and $\mu(t)$), then $s(\alpha)$ is said to be stationary and α is a geodesic.

The geodesics given in this way are clearly intrinsic and independent of any particular parametric representation of the surface.

To find the equations for geodesics, we follow the usual procedure as in the calculus of variations. Writing $f = \sqrt{(2T)}$

Where $T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$,

Then

$$s(\alpha') - s(\alpha) = \int_0^1 \{f(u + \epsilon\lambda, v + \epsilon\mu, \dot{u} + \epsilon\dot{\lambda}, \dot{v} + \epsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})\}dt$$
$$= \epsilon \int_0^1 \left(\lambda\frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} + \dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}}\right) dt + O(\epsilon^2).$$

Integrating by parts,

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}} dt = \left[\lambda\frac{\partial f}{\partial \dot{u}}\right]_0^1 - \int_0^1 \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt$$

and the first term on the right is zero because $\lambda = 0$ at $t = 0$ and $t = 1$. Similarly,

$$\int_0^1 \dot{\mu}\frac{\partial f}{\partial \dot{v}} dt = -\int_0^1 \mu\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) dt$$

and

$$s(\alpha') - s(\alpha) = \epsilon\int_0^1 (\lambda L + \mu M)dt + O(\epsilon^2)$$

Where $L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right), M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)$ ………… (2)

From the definition, therefore, $s(\alpha)$ is stationary and α is a geodesic if and only if $u(t)$ and $v(t)$ are such that $\int_0^1 (\lambda L + \mu M)dt = 0$ ....(3)

- for all admissible λ, μ, i.e. functions of class 2 in $0 \leqslant t \leqslant 1$ which satisfy $\lambda = \mu = 0$ at $t = 0$ and $t = 1$.

It will now be proved that this condition implies $L = M = 0$.

Lemma. If $g(t)$ is continuous for $0 < t < 1$ and if $\int_0^1 v(t)g(t)dt = 0$

for all admissible functions $v(t)$ as defínged above, then $g(t) = 0$. Suppose there is a t_0 between 0 and 1 such that $g(t_0) \neq 0$, say $g(t_0) > 0$. Then, since g is continuous, $g(t) > 0$ in some interval (a, b) where $0 < a < t_0 < b < 1$. Now we define $v(t)$ as follows: $v(t) = 0$ for $0 \leqslant t < a$ and for $b < t \leqslant 1$, and $v(t) = (t-a)^3(b-t)^3$ for $a \leqslant t \leqslant b$. Then $v(t)$ is admissible, and

$$\int_0^1 v(t)g(t)dt = \int_a^b v(t)g(t)dt > 0$$

since $g > 0$ and $v > 0$ for $a < t < b$. The supposition that there is a t_0 such that $g(t_0) \neq 0$ is therefore false, and the lemma is proved.

The functions L and M in equation(2) are continuous because E, F, G are assumed to be of class 1 and $u(t), v(t)$ of class 2 . The lemma can therefore be applied to equation(3), first with $\mu = 0$ and λ, L in place of v, g and then with $\lambda = 0$ and μ, M in place of v, g. It follows that equation(3) is satisfied for all admissible functions λ, μ if and only if $L = M = 0$. These, then, are differential equations for $u(t)$ and $v(t)$. They do not involve the points A and B explicitly and are therefore the same for all geodesics on the surface.

Substituting $f = \sqrt{(2T)}$, then

$$L = \frac{1}{\sqrt{(2T)}}\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{1}{\sqrt{(2T)}}\frac{\partial T}{\partial \dot{u}}\right)$$
$$= \frac{1}{\sqrt{(2T)}}\left\{\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)\right\} + \frac{1}{(2T)^2}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}},$$

with a similar expression for M. The geodesic equations are therefore

$$\left.\begin{aligned} U &\equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}} \\ V &\equiv \frac{d}{dt}\left(\frac{\partial \grave{T}}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

Where $T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$,

and the left-hand members of the equations are denoted by U and V for convenience.

The expressions U and V so defined are important in relation to any curve, whether it is a geodesic or not. They satisfy the identity $\dot{u}U + \dot{v}V = \frac{dT}{dt}$ ……….(5)

because

$$\begin{aligned} \dot{u}U + \dot{v}V &= \frac{d}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right) - \ddot{u}\frac{\partial T}{\partial \dot{u}} - \ddot{v}\frac{\partial T}{\partial \dot{v}} - \dot{u}\frac{\partial T}{\partial u} - \dot{v}\frac{\partial T}{\partial v} \\ &= \frac{d}{dt}(2T) - \frac{dT}{dt} = \frac{dT}{dt}, \end{aligned}$$

remembering that T is a function of $u, v, \dot{u}, \dot{v}$ homogeneous of degree 2 in $\dot{u}, \dot{v}$.

Since also the expressions on the right in (4) satisfy the same identity, i.e.

$$\dot{u}\left(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}\right) + \dot{v}\left(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}}\right) = \frac{1}{2T}\frac{dT}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right) = \frac{dT}{dt}$$

it follows that the two equations in (4) are not independent; they are therefore equivalent to only one equation for the two unknown functions $u(t)$ and $v(t)$.

This is to be expected because the parameter t has not been defined in any special way; the reader should verify formally that any transformation $t' = \phi(t)$, where ϕ is of class 2 , would leave the differential equations unaltered. It is convenient to regard a curve as defined by two functions $u = u(t), v = v(t)$, but strictly speaking there is only one function of one variable involved, as in the equation $v = f(u)$.

Eliminating dT/dt between the two equations (4), we obtain $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$ (6)

This is necessary for a geodesic. To prove that it is also sufficient, suppose that it is satisfied by functions $u(t)$ and $v(t)$, whose first derivatives do not vanish simultaneously at any point. Then $\partial T/\partial \dot{u}$ and $\partial T/\partial \dot{v}$ cannot vanish together since this would imply

$E\dot{u} + F\dot{v} = 0 = F\dot{u} + G\dot{v}$, and therefore $\dot{u} = \dot{v} = 0$. Hence,

$$U = \theta \frac{\partial T}{\partial \dot{u}}, V = \theta \frac{\partial T}{\partial \dot{v}}$$

for some θ, and from the identity (5),

$\frac{dT}{d\bar{t}} = \theta \left(\dot{u} \frac{\partial T}{\partial \dot{u}} + \dot{v} \frac{\partial T}{\partial \dot{v}} \right) = 2T\theta$ i.e. $\theta = (1/2T)(dT/dt)$. The functions $u(t)$ and $v(t)$ therefore satisfy equation (4).

Example 1:

Prove that the curves of the family $v^3/u^2 =$ constant are geodesics on a surface with metric

$v^2du^2 - 2uvdudv + 2u^2dv^2$ $(u > 0, v > 0)$.

Solution:

Consider $v^3/u^2 = c(> 0)$ and put this into a convenient parametric form

$u = ct^3, v = ct^2$. Then $\dot{u} = 3ct^2, \dot{v} = 2ct$ and

$$\frac{\partial T}{\partial u} = -vi\dot{v} + 2u\dot{v}^2 = 2c^3t^5, \qquad \frac{\partial T}{\partial v} = v\dot{v}^2 - u\dot{u}\dot{v} = 3c^3t^6,$$
$$\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6, \qquad \frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2u^2\dot{v} = c^3t^7,$$
$$U = \frac{d}{dt}(c^3t^6) - 2c^3t^5 = 4c^3t^5, \quad V = \frac{d}{dt}(c^3t^7) - 3c^3t^6 = 4c^3t^6.$$

Hence $V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0$, i.e. the curve is a geodesic for every value of c.

Example 2:

Prove that, on the general surface, a necessary and sufficient condition that the curve

$v = c$ be a geodesic is $EE_2 + FE_1 - 2EF_1 = 0$ ....(7)

when $v = c$, for all values of u.

On the curve $v = c, u$ can be taken as parameter, i.e. the curve is $u = t, v = c$. Then

$\dot{u} = 1, \dot{v} = 0$, and on substituting these values

(after calculating the partial derivatives of T),

$$\frac{\partial T}{\partial u} = \frac{1}{2}E_1, \quad \frac{\partial T}{\partial \dot{u}} = E, \quad U = \frac{dE}{dt} - \frac{1}{2}E_1 = \frac{1}{2}E_1,$$
$$\frac{\partial T}{\partial v} = \frac{1}{2}E_2, \quad \frac{\partial T}{\partial \dot{v}} = F, \quad V = \frac{dF}{dt} - \frac{1}{2}E_2 = F_1 - \frac{1}{2}E_2.$$

The curve is therefore a geodesic if

$$E\left(F_1 - \frac{1}{2}E_2\right) - F\left(\frac{1}{2}E_1\right) = 0.$$

when $v = c$. This is condition (7) which is therefore necessary.

Conversely when (7) is satisfied so is (6) and the curve $v = c$ is a geodesic.

If (7) is satisfied for all values of u and v, the parametric curves $v =$ constant are all geodesics.

Similarly, the curve $u = c$ is a geodesic if and only if

$$GG_1 + FG_2 - 2GF_2 = 0 \ldots\ldots\ldots..(8)$$

when $u = c$.

In the neighbourhood of a point of a geodesic at which $\dot{u} \neq 0, u$ can be taken as the parameter, as in Example 2 above. Then $\dot{u} = 1$,

$$\frac{\partial T}{\partial \dot{u}} = E + F\dot{v}, \frac{d}{du}\left(\frac{\partial T}{\partial \dot{u}}\right) = E_1 + (E_2 + F_1)\dot{v} + F_2\dot{v}^2 + F\ddot{v}$$

and

$$U = F\ddot{v} + \left(F_2 - \frac{1}{2}G_1\right)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1.$$

Also

$$\frac{\partial T}{\partial \dot{v}} = F + G\dot{v}$$

$$V = G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2$$

Hence

$$\frac{\partial T}{\partial \dot{u}}V - \frac{\partial T}{\partial \dot{v}}U = H^2(\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S)$$

where $H^2P = \frac{1}{2}(GG_1 + FG_2 - 2GF_2)$, etc. The curve $v = v(u)$ is therefore a geodesic if v satisfies a second-order differential equation of the form

$$\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0,$$

where P, Q, R, and S are functions of u and v determined by E, F, G, and their first derivatives.

This gives some idea of the complicated nature of the geodesic equation in general. A form which is more convenient for theoretical investigations will be given in the next section.

3.2. Canonical geodesic equations:

The parameter t is arbitrary and can conveniently be taken to be the arc length s of the curve measured from some fixed point on it.

(This could not be done earlier because in the variational problem the limits of the independent variable were required to be fixed.)

When there is no ambiguity a prime will denote differentiation with respect to s. Then with s as parameter, $\dot{u}, \dot{v}$ are replaced by u', v' and

$$T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2) \quad \dots\dots\dots.(1)$$

Along the curve, u' and v' satisfy the identity for direction coefficients.

Hence $T = \frac{1}{2}, dT/ds = 0,$

the canonical equations for geodesics:

$$\left.\begin{aligned} U &\equiv \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \\ V &\equiv \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \end{aligned}\right\} \ldots\ldots\ldots..(2)$$

It must be remembered that in these equations the partial derivatives of T are calculated from (1) before values for u' and v' are substituted; T is not equal to $\frac{1}{2}$ identically for all u, v, u', v', but only along the curve.

$$u'U + v'V = 0$$

confirming that equations (2) are not independent. For a curve other than a parametric curve, $u' \neq 0, v' \neq 0$, and the conditions $U = 0$ and $V = 0$ are equivalent, either being sufficient for a geodesic. For a parametric curve $u =$ constant, $u' = 0, v' \neq 0$, and $V = 0$ for all s, so that the equation is satisfied automatically; the condition for a geodesic is therefore $U = 0$. Similarly, $V = 0$ is the sufficient condition for a curve $v =$ constant to be a geodesic.

Example 1:

To find the geodesics on a surface of revolution.
Then $T = \frac{1}{2}\{(f_1^2 + g_1^2)u'^2 + g^2v'^2\}$,

where $f_1 = df/du$, etc., and since $\partial T/\partial v = 0$ the canonical equation $V = 0$ can be integrated immediately to give

$$g^2v' = \alpha$$

where α is an arbitrary constant which can be assumed non-negative, taking the positive sense along the curve to be that in which v increases. If $\alpha = 0$, then v is constant and every meridian is a geodesic. Assume now that α is positive. Then the first order differential equation can be written
giving

$$g^4dv^2 = \alpha^2ds^2 = \alpha^2\{(f_1^2 + g_1^2)du^2 + g^2dv^2\}$$
$$\alpha_{\sqrt{\prime}}(f_1^2 + g_1^2)du \pm g\sqrt{(g^2 - \alpha^2)}dv = 0$$

the $\pm$ being included although α is arbitrary because dv/du may change sign along the same geodesic. If $g^2 \neq \alpha^2$, by integration the geodesics are given by an equation of the form

$$v = \alpha\phi(u, \alpha) + \beta$$

where α, β are arbitrary constants.

If $g^2 = \alpha^2$, then $u =$ constant. However, for curves $u =$ constant the equation $V = 0$ is automatically satisfied. To see whether $u = c$ is a geodesic it is necessary to apply the condition $U = 0$. Since now $u' = 0$ and $v' = g^{-1}$ from the identity for direction coefficients,

$$\frac{\partial T}{\partial u'} = 0, \qquad \frac{\partial T}{\partial u} = \frac{g_1}{g}, \qquad U = -\frac{g_1}{g}$$

The curve $u = c$ is therefore a geodesic if and only if $g_1(c) = 0$. Since g is the radius of the parallel $u = c$ on the surface of revolution, a parallel is a geodesic if its radius is stationary.

The method used in the above example can be applied to give the following result which will be left to the reader to verify. If E, F, and G are functions of only one parameter, u say, the geodesics can all be found by quadratures. This applies not only to the general surface of revolution but also to the general helicoid. The geodesics are given by the equation

$$v = \int \left\{-\frac{F}{G} \pm \frac{\alpha H}{G(G-\alpha^2)^{1}}\right\} du + \beta$$

where α and β are arbitrary constants; and also by the equation $u = c$ where c is any root of the equation $G_1 = 0$. If F^2/E is constant, then every curve $v =$ constant is a geodesic.

Example 2:

On a right helicoid of pitch $2\pi a$, a geodesic makes an angle α with a generator at a point distant c from the axis $\left(0 < \alpha < \frac{1}{2}\pi, c > 0\right)$. Prove that the geodesic meets the axis if $c\tan\alpha < a$, but that if $c\tan\alpha > a$, its least distance from the axis is

$(c^2\sin^2\alpha - a^2\cos^2\alpha)^{\frac{1}{2}}$. Find the equation of the geodesic in the case $c\tan\alpha = a$.

From the equations metric of the right helicoid is

$du^2 + (u^2 + a^2)dv^2$. As in the above examples, a first integral of the geodesic equations is

$$\frac{dv}{du} = \frac{\pm k}{\{(u^2+a^2)(u^2+\cdot u^2-k^2)\}^3},$$

where k is an arbitrary positive constant. Further integration in general requires elliptic functions.

The given point is $(c, 0)$ for a suitable choice of axes, and α is the angle between the directions (1,0) and (u', v') at this point,

i.e. $\tan\alpha = Hv'/u' = k(c^2 + a^2 - k^2)^{-\frac{1}{2}}$. This gives $k = (c^2 + a^2)^{\frac{1}{2}}\sin\alpha$.

There are two geodesics satisfying the given initial conditions, but it will be sufficient to consider the one for which $\frac{dv}{du} < 0$ initially.

From the form of dv/du it appears that there are three cases. (i) $k^2 > a^2$, i.e. $c\tan\alpha > a$. Since $dv/du < 0$ initially, u decreases as v increases until $u = (k^2 - a^2)^{\frac{1}{2}} = (c^2\sin^2\alpha - a^2\cos^2\alpha)^{\frac{1}{2}}$.

As v continues to increase, the sign of dv/du changes and u increases indefinitely. The least distance from the axis is therefore $(c^2\sin^2\alpha - a^2\cos^2\alpha)^{\frac{1}{2}}$.

(ii) $k^2 < a^2$, i.e. $c\tan\alpha < a$. In this case $dv/du < 0$ for all v, and u decreases indefinitely as v increases. There is a point on the curve at which $u = 0$, i.e. the curve meets the axis. (iii) $k^2 = a^2$, i.e. $c\tan\alpha = a$. In this special case

$$\frac{dv}{du} = \frac{-a}{u(u^2 + a^2)^{\frac{1}{2}}}$$

and $v = -\beta + \sinh^{-1}(a/u)$ where $\beta = +\sinh^{-1}(a/c)$, since $v = 0$ when $u = c$. The geodesic is therefore given by

$$u\sinh(v + \beta) = a, \beta = \sinh^{-1}(a/c)$$

As v increases, the curve approaches the axis without reaching it. In the opposite sense, $u \to \infty$ as $v \to -\beta$, showing that the generator $v = -\beta$ is an asymptote.

Exercise:

1.Prove that for a helicoid of non-zero pitch the sections by planes containing the axis are geodesics if and only if these sections are straight lines.

3.3. Normal property of geodesics:

The geodesic equations can be expressed in terms of $\mathbf{r}(u, v)$ by means of the following identities which hold for any functions $u(t), v(t)$ of a general parameter t;

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{u}} &= \dot{\mathbf{r}} \cdot \mathbf{r}_1 \qquad \frac{\partial T}{\partial \dot{v}} = \dot{\mathbf{r}} \cdot \mathbf{r}_2 \\ U(t) &= \ddot{\mathbf{r}} \cdot \mathbf{r}_1, \quad V(t) = \ddot{\mathbf{r}} \cdot \mathbf{r}_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots(1)$$

where, as before, $T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$.

To prove these, consider the relations

$$T = \frac{1}{2}\dot{\mathbf{r}}^2, \dot{\mathbf{r}} = \mathbf{r}_1\dot{u} + \mathbf{r}_2\dot{v}$$

Then

$$\begin{aligned} \frac{\partial T}{\partial \dot{u}} &= \dot{\mathbf{r}} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial \dot{u}} = \dot{\mathbf{r}} \cdot \mathbf{r}_1 \\ \frac{\partial T}{\partial u} &= \dot{\mathbf{r}} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial u} = \dot{\mathbf{r}} \cdot (\mathbf{r}_{11}\dot{u} + \mathbf{r}_{21}\dot{v}) = \dot{\mathbf{r}} \cdot \frac{d}{dt}(\mathbf{r}_1), \\ U(t) &= \frac{d}{dt}(\dot{\mathbf{r}} \cdot \mathbf{r}_1) - \dot{\mathbf{r}} \cdot \frac{d}{dt}(\mathbf{r}_1) = \ddot{\mathbf{r}} \cdot \mathbf{r}_1 \end{aligned}$$

and similarly for $\partial T/\partial \dot{v}$ and $V(t)$.

With s as parameter the geodesic equations are $U(s) = 0; V(s) = 0$. They can therefore be written $\mathbf{r}'' \cdot \mathbf{r}_1 = 0, \mathbf{r}'' \cdot \mathbf{r}_2 = 0$ …………(2)

showing that, at every point P of the geodesic, $\mathbf{r}''$ is perpendicular to the tangent plane at P. This condition is sufficient as well as necessary. Hence:

A characteristic property of a geodesic is that at every point its principal normal is normal to the surface. Every curve having this property is a geodesic.

In terms of a general parameter t, equation (6) can be written i.e. $(\dot{\mathbf{r}}.\mathbf{r}_1)(\ddot{\mathbf{r}} \cdot \mathbf{r}_2) - (\dot{\mathbf{r}} \cdot \mathbf{r}_2)(\ddot{\mathbf{r}} \cdot \mathbf{r}_1) = 0,$

This says that the binomial of the curve is perpendicular to the normal to the surface, from which it follows that the principal normal is normal to the surface.

An equivalent statement of the above normal property is that at every point of a geodesic the rectifying plane is tangent to the surface.

The above property often makes it possible to intuit that a curve is a geodesic. For example, every great circle of a sphere and every meridian of a surface of revolution clearly have the normal property of geodesics. Again, it is now clear that the only parallels of a surface of revolution which are geodesics are those whose radii have stationary lengths.

Example 1:

A particle is constrained to move on a smooth surface under no force except the normal reaction. Prove that its path is a geodesic.

The acceleration is in the direction $\ddot{\mathbf{r}}$ which is therefore in the direction of the force, i.e. normal to the surface. Since $\dot{\mathbf{r}}$ is tangent

to the surface, $\dot{\mathbf{r}}.\ddot{\mathbf{r}} = 0$ and $\dot{s} = |\dot{\mathbf{r}}| =$ constant, showing that the speed is constant. It follows that $\mathbf{r}''$ is in the direction $\ddot{\mathbf{r}}$, i.e. is normal to the surface, and the curve is therefore a geodesic.

This problem can also be solved by using the Lagrange equation of dynamics, taking u and v as generalized coordinates.

Exercise:

Prove that every helix on a cylinder is a geodesic.

The normal property is sometimes taken as the definition of a geodesic. It has the advantage of simplicity but obscures the intrinsic character of geodesics and could not apply to Riemannian geometry which is similar to the intrinsic geometry of a surface but with any number of dimensions. Also, the normal property strictly fails in the case of a straight line on a surface, for then the principal normal is indeterminate. Such a line is clearly a geodesic according to the intrinsic definition.

It is instructive to see how the differential equations for geodesics arise out of equations (12), and to see how certain Christoffel symbols arise at this stage, although they will arise in different contexts later in the book.

Differentiating $\mathbf{r}' = \mathbf{r}_1 u' + \mathbf{r}_2 v'$, we find

$$\mathbf{r}'' = \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11} u'^2 + 2\mathbf{r}_{12} u'v' + \mathbf{r}_{22} v'^2$$

The geodesic equations $\mathbf{r}'' \cdot \mathbf{r}_1 = 0$ and $\mathbf{r}'' \cdot \mathbf{r}_2 = 0$ thus become where $\left.\begin{array}{l} Eu'' + Fv'' + \Gamma_{111} u'^2 + 2\Gamma_{112} u'v' + \Gamma_{122} v'^2 = 0 \\ Fu'' + Gv'' + \Gamma_{211} u'^2 + 2\Gamma_{212} u'v' + \Gamma_{222} v'^2 = 0 \end{array}\right\},$ ....(3)

The coefficients Γ_{ijk} are called Christoffel symbols of the first kind and can be expressed in terms of first derivatives of the fundamental coefficients. It can easily be verified that

$$\frac{1}{2}\left\{(\mathbf{r}_i \cdot \mathbf{r}_j)_k + (\mathbf{r}_i \cdot \mathbf{r}_k)_j - (\mathbf{r}_j \cdot \mathbf{r}_k)_i\right\} = \mathbf{r}_i \cdot \mathbf{r}_{jk} = \Gamma_{ijk} \quad \text{.............. (4)}$$

Thus $\left.\begin{array}{l} \Gamma_{111} = \frac{1}{2}E_1, \Gamma_{112} = \Gamma_{121} = \frac{1}{2}E_2, \Gamma_{122} = F_2 - \frac{1}{2}G_1 \\ \Gamma_{211} = F_1 - \frac{1}{2}E_2, \Gamma_{212} = \Gamma_{221} = \frac{1}{2}G_1, \Gamma_{222} = \frac{1}{2}G_2 \end{array}\right\}$ ... ... (5)

Since $EG - F^2 \neq 0$, equations can be solved for u'' and v''. The resulting equations, which are equivalent to (12.4), are written

$$\left.\begin{array}{l} u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 = 0 \\ v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 = 0 \end{array}\right\} \text{...........(6)}$$

where the coefficients Γ_{jk}^i, called the Christoffel symbols of the secend kind, are given by

$$\Gamma_{jk}^1 = H^{-2}(G\Gamma_{1jk} - F\Gamma_{2jk}), \Gamma_{jk}^2 = H^{-2}(E\Gamma_{2jk} - F\Gamma_{1jk}) \text{ ... ... (7)}$$

3.4. Existence theorems:

With s as parameter the geodesic equations can be written in the form

$$u'' = f(u, v, u', v'), v'' = g(u, v, u', v') \text{ ....(1)}$$

where f and g are quadratic forms in u', v' with single-valued continuous functions of u and v as coefficients. These are simultaneous second order differential equations for u and v as functions of s, and from the theory of such equations, † if f and g are of class $\geqslant 1$, a solution exists and is determined uniquely by arbitrary initial values of u, v, u', and v'. Hence:

A geodesic can be found to pass through any given point and have any given direction at that point. The geodesic is determined uniquely by these initial conditions.

From the above existence theorem it is to be expected that if a point Q is sufficiently near any given point P, then it is possible to find a direction at P such that the geodesic through P in this direction also passes through Q. The following theorem can in fact be proved, assuming merely that the surface is of class 3.

Every point P of the surface has a neighbourhood N with the property that every point of N can be joined to P by a unique geodesic arc which lies wholly in N.

This does not, of course, state that if Q is a point of N then the geodesic arc PQ which lies in N is the only geodesic joining P and Q; there may be other geodesic arcs PQ but they leave N. Examples of this will be given later in this section.

This theorem gives all that we can say at present about the existence of geodesics joining two given points; it says that Q can be joined to P if it is sufficiently near P. Nothing more than that can be said as long as the region of the surface being considered is arbitrary.

Later, however, when a complete surface has been defined, it will appear that any two points can be joined by at least one geodesic.

A region R is convex if any two points of it can be joined by a geodesic arc lying wholly in R, and is simple if there is not more than one such geodesic arc. In the Euclidean plane a convex region is necessarily simple but this is not so for a surface in general. The surface of a sphere, for example, is convex but not simple.

Every point P of a surface has a neighborhood which is convex and simple.

The difference between this and the previous theorem is that it is no longer just one particular point which is joined to the others of the neighborhood; every point is joined uniquely to every other point. Whitehead's theorem is in fact much deeper than the previous theorem and its proof is beyond the scope of this book. It will not be used in the sequel.

A particular and interesting form of Whitehead's theorem is concerned with a geodesic disk of given center P and radius r, defined as the set of points Q such that there is a geodesic arc PQ of length not greater than r. Whitehead proved that for every point P there is an $\epsilon > 0$ such that every geodesic disk of centre P and radius $r \leqslant \epsilon$ is convex and simple.

Exercise:

On a circular cylinder of radius a, find the least upper bound for the radius of a simple convex geodesic disk, and prove that a geodesic disk of greater radius is convex but not simple.

This section will be concluded with examples of the multiplicity of geodesics joining two points. They are mostly constructed by using the intrinsic property of geodesics, that if surfaces S and S' are isometric, then the curve on S' which corresponds to a geodesic on S is a geodesic on S'. In fact, the correspondence need only be locally isometric since a curve is a geodesic if every small arc is a geodesic arc.

Consider, for example, the mapping of a plane on a circular cylinder obtained by wrapping the plane round the cylinder. A geodesic on the plane is a straight line, and this corresponds to a helix (or meridian or circular section) on the cylinder. The helix is therefore a geodesic on the cylinder.

Conversely, every helix on the cylinder corresponds to a straight line (or strictly to a family of parallel straight lines) on the plane; thus every helix is a geodesic.

It follows at once that any two points P, Q of the cylinder, not on the same parallel, are joined by infinitely many geodesic arcs because there are infinitely many helices joining the two points. When the cylinder is unrolled into a plane there are infinitely many images of Q, and the geodesics PQ correspond to the straight lines joining all the images of Q to any one image of P. There is a geodesic arc PQ making any desired number of turns round the cylinder, in either sense.

A similar result holds for the anchor ring. Joining two points P, Q not on the same meridian there are infinitely many geodesic arcs; an arc can be found to make any number of turns, in either sense, of the kind made by the meridian circles, and at the same time any number of turns, in either sense, of the kind made by the parallel circles. This cannot be proved by the simple method used for the cylinder because the anchor ring is not locally isometric to a plane.

The geodesic equations can, however, be integrated by the method of section 3.2.

An example of a surface on which the number of geodesics joining two given points may be more than one but is strictly limited is a right circular half-cone. Here again the different geodesic arcs PQ are obtained by taking different numbers of turns round the cone in either sense.

A local isometry can be set up by rolling the cone over the plane. The surface of the cone corresponds isometrically to a sector of the plane which is reproduced according to the number of revolutions of the cone, in either sense. The images of Q are points $Q_1, Q_2, \ldots$ in one sense and $Q_{-1}, Q_{-2}, \ldots$ in the other. If P' is the first image of P (in the sector between Q_1 and Q_{-1}), then the geodesic arcs PQ which pass round the cone in one sense correspond to the straight lines $P'Q_1, P'Q_2, \ldots$, and those which pass round the cone in the opposite sense correspond to the lines $P'Q_{-1}, P'Q_{-2}, \ldots$. In either sense the number is limited; the lines $P'Q_r$ must all be on one side of $P'V$ and the lines $P'Q_{-r}$ must all be on the other side of $P'V$. In Fig. 2 there are three geodesic arcs in one sense and two in the other.

Clearly, the smaller the solid angle of the cone the greater the number of geodesics. Fig. 3 illustrates the case when there is only one geodesic arc PQ.

It is interesting to note that, on the cone, there may be nontrivial geodesic arcs joining a point to itself; in the above argument P can coincide with Q.

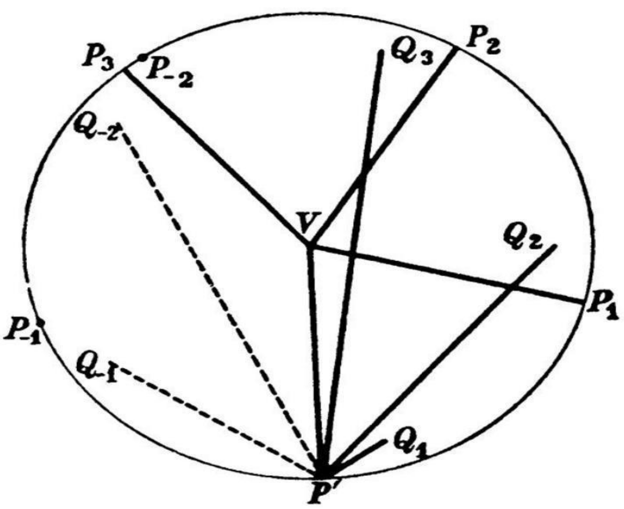

Figure. 2

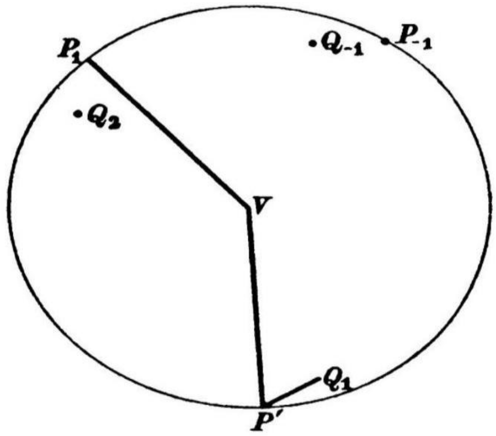

Figure. 3

With the cone as a guide it is not difficult to construct other surfaces on which there may be any finite number of geodesics joining two points, or joining a point to itself. An example is the paraboloid of revolution, on which the geodesic equation can be integrated by the method. Again, on the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$$

or on one sheet of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

the larger c is in comparison with a and b the more geodesics there are joining two points.

Exercise:

1.Prove that, on a right circular cone of semi vertical angle α, every point can be joined to itself by a geodesic arc if $\alpha<\frac{1}{6}\pi$. If this condition is satisfied prove that the number of geodesic arcs joining a point to itself is the greatest integer less than $(2\sin\alpha)^{-1}$. Prove also that this is the number of times a geodesic other than a generator intersects itself.

2. Prove that, on a paraboloid of revolution, every geodesic other than a meridian intersects itself infinitely often.

3.5. Geodesic parallels:

Suppose a family of geodesics is given, and that a parameter system is chosen so that the geodesics of the family are the curves $v =$ constant and their orthogonal trajectories are the curves $u =$ constant. Then $F = 0$ and condition for the curves $v =$ constant to be geodesics becomes $E_2 = 0$. The metric is therefore of the form

$$ds^2 = E(u)du^2 + G(u,v)dv^2 \quad \text{........(1)}$$

Consider the distance between any two of the orthogonal trajectories, say $u = u_1$ and $u = u_2$, measured along the geodesic $v = c$. Along $v = c, dv = 0$, and $ds = \sqrt{E}du$, so that the distance is

$$\int_{u_1}^{u_2} \sqrt{E}(u)du,$$

a number independent of c. The distance is thus the same along whichever geodesic $v =$ constant it is measured. Because of this, the orthogonal trajectories are called geodesic parallels.

In the plane, a family of geodesics is a family of straight lines enveloping some curve C, and the geodesic parallels are the involutes of C. In particular, when the geodesics are concurrent straight lines, the parallels are concentric circles.

In the above metric the parameter u can be specialized by taking it to be the distance from some fixed parallel to the parallel determined by u, the distance being measured along any geodesic $v = c$. Then $ds = du$ when $dv = 0$, i.e. $E = 1$. Hence: for any given family of geodesics, a parameter system can be chosen so that the metric takes the form $du^2 + Gdv^2$. The given geodesics are the parametric curves $v =$ constant and their orthogonal trajectories are $u =$ constant, u being the distance measured along a geodesic from some fixed parallel.

The transformation $u \to u': du' = \sqrt{E}du$ also gives the simplified metric from (1).

Exercise:

If a surface admits two orthogonal families of geodesics, it is isometric with the plane.

Geodesic polar

A particularly useful system of geodesics and parallels is found by taking the geodesics which pass through a given point 0. By the second existence theorem there is a neighborhood of O in which, when the point O itself is excluded, the geodesics constitute a family. Parameters u, v can therefore be chosen as above. In particular u can be taken to be the distance measured from O along the geodesics and v can be taken to be the angle measured at O between a fixed geodesic $v = 0$ and the one determined by v. In this way u and v correspond to polar coordinates r and θ in the plane. The metric is therefore

$$du^2 + Gdv^2$$

where G is such that, when u is small, the metric approximates to the plane polar form with u, v in place of r, θ, i.e. to $du^2 + u^2dv^2$. Hence $G \sim u^2$, i.e.

$$\lim_{u \to 0} \frac{\sqrt{G}}{u} = 1$$

In geodesic polar parameters the parallels $u =$ constant are geodesic circles.

3.6. Geodesic curvature:

For any curve on a surface the curvature vector at a point P is $\mathbf{r}'' = \kappa\mathbf{n}$, where κ is the curvature and $\mathbf{n}$ is the principal normal. This can be written

$$\mathbf{r}'' = \kappa_n\mathbf{N} + \lambda\mathbf{r}_1 + \mu\mathbf{r}_2 \quad \ldots\ldots\ldots..(1)$$

where κ_n is the normal component of $\mathbf{r}''$, called the normal curvature. The vector $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$, with components (λ, μ), is zero for a geodesic because then r'' is normal to the surface. This suggests that for any curve the vector (λ, μ) is intrinsic so that its magnitude measures in some sense the deviation of the curve from a geodesic. The vector (λ, μ) is, in fact, intrinsic, for from (20), taking scalar products with r_1 and $\mathbf{r}_2$,

$$E\lambda + F\mu = \mathbf{r}'' \cdot \mathbf{r}_1 = U, F\lambda + G\mu = \mathbf{r}'' \cdot \mathbf{r}_2 = V \quad \ldots\ldots\ldots(2)$$

where U and V are calculated with s as parameter. Thus λ and μ are given by the intrinsic formulae

$\lambda = H^{-2}(GU - FV), \mu = H^{-2}(EV - FU).$ .... (3)

The vector (λ, μ) is called the geodesic curvature vector of the curve under consideration. In the notation introduced at the end of section 3.3 the components λ, μ are given by

$$\lambda = u'' + \Gamma_{11}^{1} u'^2 + 2\Gamma_{12}^{1} u'v' + \Gamma_{22}^{1} v'^2,$$

The geodesic curvature vector of any curve is orthogonal to the curve. This follows at once from (20) since the tangent vector $\mathbf{r}'$ is orthogonal to $\mathbf{r}''$ and to $\mathbf{N}$ and therefore also to $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$, which is the geodesic curvature vector. It can also be proved intrinsically; the orthogonality condition for the vectors (u', v') and (λ, μ) can be written

$$u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0$$

which from (21) becomes the identity $u'U + v'V = 0$.

Exercise:

Prove that the components λ, μ of the geodesic curvature vector are given by the following formulae, with s as parameter.

$$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}, \mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = -\frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'}.$$

The geodesic curvature, κ_g, of any curve is defined as the magnitude of the geodesic curvature vector with a sign attached, positive or negative according as the angle between the tangent and the geodesic curvature vector is $+\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$. The geodesic curvature is therefore intrinsic. From the sine formula for the angle between the vectors (u', v') and (λ, μ) it follows that $\kappa_g = H(u'\mu - v'\lambda)$

The geodesic curvature of a geodesic is zero. Conversely, a curve with zero geodesic curvature at every point has zero geodesic curvature vector and is therefore a geodesic.

Since the unit tangent vector $\mathbf{r}'$ is orthogonal to $\mathbf{N}$, the unit vector which lies in the tangent plane and makes an angle $+\frac{1}{2}\pi$ with $\mathbf{r}'$ is $\mathbf{N} \times \mathbf{r}'$. The geodesic curvature vector is therefore $\kappa_g \mathbf{N} \times \mathbf{r}'$, and (20) can be written $\mathbf{r}'' = \kappa_n \mathbf{N} + \kappa_g \mathbf{N} \times \mathbf{r}'$

Taking the scalar product with the unit vector $\mathbf{N} \times \mathbf{r}'$, we have $\kappa_g = [\mathbf{N}, \mathbf{r}', \mathbf{r}'']$

In this formula for κ_g it is a simple matter to pass from s to a general parameter t. Since $\mathbf{r}' = \dot{\mathbf{r}}/\dot{s}$ and $\mathbf{r}' \times \mathbf{r}'' = \dot{\mathbf{r}} \times \ddot{\mathbf{r}}/\dot{s}^3$ the formula becomes $\kappa_g = \dot{s}^{-3}[\mathbf{N}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}]$

Substituting $\mathbf{N} = H^{-1}\mathbf{r}_1 \times \mathbf{r}_2$, we have

$$\kappa_g = H^{-1}\dot{s}^{-3}(\mathbf{r}_1 \times \mathbf{r}_2) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})$$
$$= H^{-1}\dot{s}^{-3}\{(\mathbf{r}_1 \cdot \dot{\mathbf{r}})(\mathbf{r}_2 \cdot \ddot{\mathbf{r}}) - (\mathbf{r}_2 . \dot{\mathbf{r}})(\mathbf{r}_1 . \ddot{\mathbf{r}})\}$$

and because of the identities (12.1) this can be written $\kappa_g = \frac{1}{H\dot{s}^3}\left(\frac{\partial T}{\partial \dot{u}} V(t) - \frac{\partial T}{\partial \dot{v}} U(t)\right)$

Example 1:

To find the geodesic curvature of the parametric curve $v = c$. Taking u as parameter, then $\dot{u} = 1, \dot{v} = 0$, and

$. \frac{\partial T}{\partial \dot{u}} = E, \frac{\partial T}{\partial \dot{v}} = F, U = \frac{1}{2}E_1, V = F_1 - \frac{1}{2}E_2.$

Also, $\dot{s} = E^{\ddagger}$. Hence the required curvature is given by

$$\kappa_g = \frac{1}{2}H^{-1}E^{-\mathbf{1}}(2EF_1 - EE_2 - FE_1).$$

$t = s$ can be simplified by means of the identity $u'U(s) + v'V(s) = 0$. Substituting for either V or U and using the fact that $u'(\partial T/\partial u') + v'(\partial T/\partial v') = 2T = 1$ when s is the parameter, $\kappa_g = -\frac{1}{H}\frac{U(s)}{v'} = \frac{1}{H}\frac{V(s)}{u'}$

Exercise:

Prove that if (λ, μ) is the geodesic curvature vector, then $\kappa_g = \frac{-H\lambda}{Fu'+Gv'} = \frac{H\mu}{Eu'+H'v'}$

Geodesic curvature may be regarded as the intrinsic generalization of curvature of plane curves, as can be seen from the following result which will not be proved here.

Let P be a point of a given curve C on a surface and Q the voint of C at a distance δs from P along C. If the geodesics wnuch are tangent to C at P and Q meet at the point R, let $\delta\psi$ be the angle between the tangents to these geodesics at R. Then the geodesic curvature of C at P is

$\lim_{\delta s \to 0} \frac{\delta \psi}{\delta s}$.

For a plane curve, $\delta\psi$ is the angle between the tangents at P and Q and $\lim_{\delta s \to 0} \frac{\delta \psi}{\delta s}$ is the curvature $\frac{d\psi}{ds}$ in the usual notation.

The above would be a satisfactory intrinsic definition of geodesic curvature except for the difficulty of proving that the tangent geodesics at P and Q do in fact meet at a point, R near P. A more straightforward intrinsic generalization of curvature is as follows.

Let P be a point of a given curve C on a surface and Q the point of C at a distance δ s from P along C. Let $\bar{C}$ be the geodesic arc PQ, of length $\delta\bar{s}$. Then if $\delta\theta$ is the angle between C and $\bar{C}$ at P and if $\delta\phi$ is the angle between $\bar{C}$ and C at Q, the geodesic curvature of C at P is $\lim_{\delta s \to 0} \frac{\delta\theta + \delta\phi}{\delta s}$ (see Fig. 4 ; note that for this figure κ_0 is negative).

There is no difficulty about this construction because of the existence theorem for a geodesic joining two neighboring points. To prove the result, let $(u', v'), (u_0', v_0')$ be unit tangent vectors to C at Q and P respectively. Let $(\bar{u}', \bar{v}'), (\bar{u}_0', \bar{v}_0')$ be unit tangent vectors to $\bar{C}$ at Q and P respectively. Then

$$\sin\delta\theta = H(u_0, v_0)\{u_0'\bar{v}_0' - v_0'u_0'\}, \sin\delta\phi = H(u,v)\{\bar{u}'v' - u'\bar{v}'\}.$$

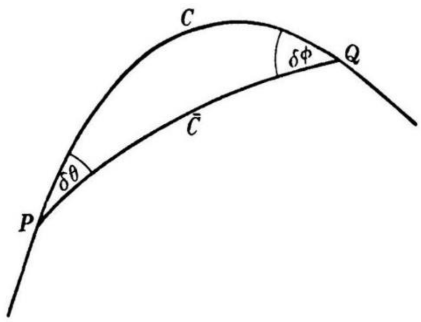

Figure. 4

We have

$$u' = u_0' + \delta s u_0'' + O(\delta s^2) \text{ as } \delta s \to 0,$$
$$v' = v_0' + \delta s v_0'' + O(\delta s^2) \text{ as } \delta s \to 0,$$
$$\bar{u}' = \bar{u}_0' + \delta\bar{s} f(u_0, v_0, \bar{u}_0', \bar{v}_0') + O(\delta\bar{s}^2) \text{ as } \delta\bar{s} \to 0,$$
$$\bar{v}' = \bar{v}_0' + \delta\bar{s} g(u_0, v_0, \bar{u}_0', \bar{v}_0') + O(\delta\bar{s}^2) \text{ as } \delta\bar{s} \to 0,$$

where in the last two equations we have used the geodesic equations. Also we have

$$\delta\bar{s} = \delta s + O(\delta s^2) \text{ as } \delta s \to 0$$

We write

$$H(u_0, v_0) = H_0, H(u, v) = H_0 + \delta H$$

Where $\delta H = O(\delta s)$ as $\delta s \to 0$

Then we have $\sin\delta\theta + \sin\delta\phi$

$$= H_0\delta s[u_0'\{v_0'' - g(u_0, v_0, \bar{u}_0', \bar{v}_0')\} - v_0'\{u_0'' - f(u_0, v_0, \bar{u}_0', \bar{u}_0')\}] + \\ + \\ \delta H(\bar{u}_0' v_0' - \bar{v}_0' u_0') + O(\delta s^2)$$

Also, as $\delta s \to 0$ we have

$$\sin\delta\theta = \delta\theta + O(\delta s^2), \qquad \sin\delta\phi = \delta\phi + O(\delta s^2)$$
$$\bar{u}_0' = u_0' + O(\delta s), \qquad \bar{v}_0' = v_0' + O(\delta s)$$

Then

$$\delta\theta + \delta\phi = H_0\delta s[u_0'\{v_0'' - g(u_0, v_0, u_0', v_0')\} - \\ -v_0'\{\{ {}_0'' - f(u_0, v_0, u_0', v_0')\} + O(\delta s^2) \text{ as } \delta s \to 0.$$

From (15.4), the geodesic curvature vector (λ, μ) of C at P is given by

$$\lambda = u_0'' - f(u_0, v_0, u_0', v_0'), \mu = v_0'' - g(u_0, v_0, u_0', v_0').$$

Hence $\frac{\delta\theta + \delta\phi}{\delta s} = H_0(u_0'\mu - v_0'\lambda) + O(\delta s).$

Thus, proceeding to the limit as $\delta s \to 0$ and dropping the suffix, we get

$$\lim_{\delta s \to 0} \frac{\delta\theta + \delta\phi}{\delta s} = H(u'\mu - v'\lambda) = \kappa_g$$

Liouville's formula for κ_g. This is an expression for κ_g involving the angle θ which the curve under consideration makes with the parametric curves $v =$ constant. Regarding θ as a function of s along the curve, then Liouville's formula is

$\kappa_g = \theta' + Pu' + Qv'$

where

$$P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2), Q = \frac{1}{2HE}(EG_1 - FE_2).$$

The direction coefficients of the curve $v =$ constant and the given curve are $(1/\sqrt{E}, 0)$ and (u', v') so that

$$\cos\theta = \sqrt{E}u' + \frac{F}{\sqrt{E}}v' = \frac{1}{\sqrt{E}}\frac{\partial T}{\partial u'}, \sin\theta = \frac{H}{\sqrt{E}}v'$$

Differentiating $\cos\theta$,

$$-\sin\theta\frac{d\theta}{ds} = \frac{1}{\sqrt{B'}}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{1}{2E^{1}}(E_1u' + E_2v')\frac{\partial T}{\partial u'};$$

multiplying by $\sqrt{E}$ and substituting

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) = U + \frac{\partial T}{\partial u}$$

$$-Hv'\theta' = U + \frac{1}{2}(E_1u'^2 + 2F_1u'v' + G_1v'^2) -$$

$$-\frac{1}{2E}(E_1u' + E_2v')(Eu' + Fv')$$

$$= U + \frac{1}{2E}\{(2EF_1 - FE_1 - EE_2)u'v' + (EG_1 - FE_2)v'^2\}.$$

Lioùville's formula now appears on dividing by Hv' and substituting $U = -\kappa_g Hv'$ from (1).

Example 2:

Prove that if the orthogonal trajectories of the curves $v =$ constant are geodesics, then H^2/E is independent of u.

The orthogonal trajectories satisfy $\theta = \frac{1}{2}\pi$ and are geodesics if $\kappa_g = 0$. From Liouville's formula, $Pu' + Qv' = 0$. Also $\cos\theta = 0$, i.e. $Eu' + Fv' = 0$, and the trajectories will be geodesics if

$EQ - FP = 0$

On substituting for P and Q the condition becomes

$F^2E_1 - 2EFF_1 + E^2G_1 = 0,$

i.e. $\partial(G - F^2/E)/\partial u = 0$ as required.

Exercise:

1.Prove that if a curve C on a surface is projected orthogonally on to the tangent plane at a point P of C, it becomes a plane curve whose curvature at P is the geodesic curvature of C at P.

3.7. Gauss-Bonnet theorem:

Consider a surface of class 3 , with parameter system u, v, and let a closed curve C be the boundary of a simply connected region $\boldsymbol{R}$ of the surface. (By simply connected we mean that every closed curve lying in R can be contracted continuously into a point without leaving R.) Suppose that C consists of n arcs

$A_0A_1, A_1A_2, \dots, A_{n-1}A_n \ (A_n = A_0)$

where n is finite, and that each arc is of class 2 . The vertices A_0, A_1, ... are taken in order along C to agree with the positive sense of description of C; this is usually described as the sense which 'leaves the interior on the left', i.e. a positive rotation of $\frac{1}{2}\pi$ from the tangent gives the normal which points to the interior region R. At the vertex $A_r (r = 1, \dots, n)$ let α_r be the angle between the tangents to the arcs $A_{r-1}A_r$ and A_rA_{r+1}, measured with the usual convention at A_r so that $-\pi < \alpha_r < \pi$; at A_n, α_n is the angle between the tangents to $A_{n-1}A_n$ and A_nA_1. Regarding C as a 'curvilinear polygon', $\alpha_1, \dots, \alpha_n$ are the exterior angles at the vertices $A_1, \dots, A_n$ (see Fig. 5 where $n = 6$).

The geodesic curvature exists at every point of C except possibly at the vertices, and the line integral $\int_C \kappa_g ds$ can therefore be calculated. The excess of C is defined as

$$\text{ex}C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_C \kappa_g ds$$

This is an invariant, independent of the particular parameter system for the surface. The only possible effect of a change of parameter system is to reverse at every point the sense in which angles are measured; this would reverse the sense along C and therefore the description of the polygon, but κ_g at each point and α_r at each vertex would remain unchanged.

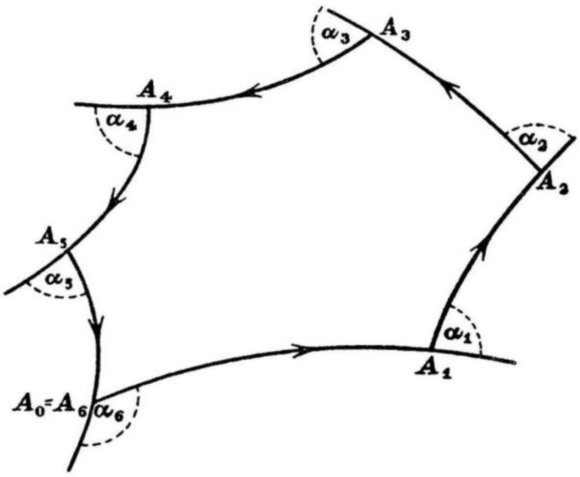

Fig. 5

For a curvilinear polygon C on the plane, κ_g is the ordinary curvature $d\psi/ds$ and $\int \kappa_g ds + \sum_{r=1}^{n} \alpha_r$ is the total angle through which the tangent turns in describing C. This angle is clearly 2π, so that the excess of C is zero. In particular, for a rectilinear polygon, $\kappa_g = 0$ at every point and $\sum \alpha_r$ is the sum of the exterior angles, i.e. 2π, giving ex $C = 0$. Since excess, as defined above, is intrinsic, it follows, that on any surface isometric with the plane, the excess of a simple closed curve is zero.

This result suggests that for a surface which is not isometric with the plane, the excess of a simple curve C enclosing a region R is in some sense a measure of the intrinsic difference between R and a region of the plane. The excess may therefore lead to an intrinsic definition of the curvature of a surface, based on the convention that a plane has zero curvature. This is in fact the case, and it will be shown that from the excess can be derived the important invariant known as the Gaussian curvature of a surface.

From Liouville's formula for κ_g,

$$\int_C \kappa_g ds = \int_C (d\theta + Pdu + Qdv),$$

where θ is the angle which C makes with the parametric curve $v =$ constant and P and Q are certain functions of u and v. Since the curves $v =$ constant form a family in the region R bounded by C, the tangent to C turns through 2π relative to these curves, i.e.

Hence

$$\int_C d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi$$

$$\text{ex } C = -\int_C (Pdu + Qdv).$$

By Green's theorem, since R is simply connected and P and Q are differentiable functions of u and v in R,

$$\int_{\boldsymbol{C}} (Pdu + Q_1 dv) = \int_{\boldsymbol{R}} \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dudv.$$

Hence, writing $dS = Hdudv$ for the surface element, ex$C = \int_{\boldsymbol{K}} KdS$ ……….(1)

where K is a function of u and v, independent of the curve C, given by

$K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$ ……….. (2)

Equation (1) shows that there is a certain function K of u and v which is determined by E, F, and G, and that the excess of any curve C which encloses a simply connected region R is equal to the surface integral of K over R. We shall now show that the function K is uniquely determined. Let $\bar{K}$ be a second function which also satisfies (1) and is independent of C. Then for every region R,

$\int_R (\bar{K} - K) dS = 0$ ………. (3)

Now suppose $\bar{K} \neq K$ at some point P, say $\bar{K} > K$. Then since $\bar{K} - K$ is continuous, there is a region R which contains P and in which $\bar{K} - K > 0$ at every point. For this R, $\int_R (\bar{K} - K) dS > 0$ which contradicts (3). A similar contradiction exists if $\bar{K} < K$ at P. Hence $\bar{K} = K$ at every point, i.e. K is uniquely determined as a function of u and v.

From this uniqueness property and from the form of (1) it follows that K is an invariant; at every point the value of K is independent of the parameter system. Also K is intrinsic, since it can be calculated when the metric is known. Thus K is an intrinsic geometrical invariant; it is called the Gaussian curvature of the surface.

For any region R, whether simply connected or not, $\int_R KdS$ is called the total curvature of R. Equation (1) now gives the Gauss-Bonnet theorem. For any curve C which encloses a simply connected region R, the excess of C is equal to the total curvature of R. For a geodesic triangle ABC, formed by geodesic ares AB, BC, CA and enclosing a simply connected region R, the excess is

$$2\pi - (\pi - A) - (\pi - B) - (\pi - C) = A + B + C - \pi,$$

where A, B, C are the interior angles of the triangle. Thus the excess is the excess of $A + B + C$ over its Euclidean value π, a fact which accounts historically for our use of the word 'excess'. The total curvature of a geodesic triangle ABC is therefore equal to $A + B + C - \pi$. More generally, for a geodesic polygon of any number of sides (geodesic arcs) the total curvature is equal to 2π minus the sum of the exterior angles, i.e. the excess of the sum of the interior angles over $(n - 2)\pi$ where n is the number of sides.

Exercise:

By first considering the region of the anchor ring of section 3 bounded by two meridians and the two parallels $u = 0, u = \pi$, prove that the total curvature of the whole surface is zero.

3.8. Gaussian curvature:

An historical definition of Gaussian curvature K follows from the Gauss-Bonnet theorem for a geodesic triangle. If P is a given point and Δ the area of a geodesic triangle ABC which contains P, then at P, $K = \lim \frac{A+B+C-\pi}{\Delta}$ …………(1)

where the limit is taken as all vertices tend to P. On a sphere of radius a, for example, the geodesics are great circles, and the area of a geodesic triangle ABC is $a^2(A + B + C - \pi)$. The Gaussian curvature at every point is therefore $1/a^2$.

That K is constant over the sphere is to be expected from the fact that there is an isometric mapping of the sphere on itself in which any given point P corresponds to any other given point Q, so that $(K)_P = (K)_Q$ since K is an intrinsic invariant.

The formula $K = 1/a^2$ at a point of a sphere of radius a illustrates the fact that the dimensions of K are (length) $^{-2}$. This follows more generally from the Gauso-Bonnet equation, in which the excess of a curve is clearly dimensionless.

The total curvature $\int_R KdS$ for any region R is dimensionless. On a sphere of radius a, for example, the total curvature for the whole sphere is area $/a^2 = 4\pi$. It will be seen in a later chapter that the total curvature of a compact surface depends only upon the topology of the surface.

The formula for K in terms of E, F, and G is given by (16.2), where P and Q are given by (15.11). Hence, at any point and in any parameter system,

$$K = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2 - EG_1}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right) \quad \ldots\ldots\ldots\ldots (2)$$

When the parametric curves are orthogonal, $F = 0$ and the formula for K can be written if the simpler and symmetric form $K = -\frac{1}{2H}\left\{\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right\}$ ....(3)

where now $H = \sqrt{(EG)}$

For example, on a sphere of radius a parameters can be chosen as in section 3 so that $E = a^2, F = 0, G = a^2 \sin^2 u$. Then $H = a^2 \sin u$ since $0 < u < \pi$, and the above formula gives

$$K = -\frac{1}{2a^2 \sin u}\frac{\partial}{\partial u}(2\cos u) = \frac{1}{a^2}.$$

In Chapter III a very different kind of formula (non-intrinsic) for K will be given in terms of the second fundamental coefficients to be defined later. This formula is appropriate when the position vector $\mathrm{r}(u, v)$ for the surface is given and is generaily simpler than the above for the purpose of calculation. It cannot, however, be applied when only the metric is given.

Exercise:

1.Find the Gaussian curvature at the point (u, v) of the anchor ring of section 3 and verify that the total curvature of the whole surface is zero.

2. Prove that the Gaussian curvature of the surface given (in Monge form) by $z = f(x, y)$ is $(rt - s^2)(1 + p^2 + q^2)^{-2}$, where $p, q, r, s,$ and t denote respectively

$\partial z/\partial x, \partial z/\partial y, \partial^2 z/\partial x^2, \partial^2 z/\partial x \partial y,$ and $\partial^2 z/\partial y^2$.

Geodesic polar form

With geodesic polar parameters the metric takes the form $du^2 + g^2 dv^2$

where for convenience g is written for $\sqrt{G}$. In section 14 it was shown that $g(u, v)$ satisfies the condition $g = u + O(u^2)$ as $u \to 0$.

The Gaussian curvature at the point (u, v) is given by (3) with $E = 1, G = g^2$, and $H = g$. Hence $K = -g_{11}/g$

The center (origin) of the geodesic polar parameters is excluded from the domain of u, v because it is a singularity, but since this is only artificial the Gaussian curvature exists there; suppose it has the value K_0 at the origin. Then as $u \to 0, g_{11} \sim -K_0 g \sim -K_0 u$; on integrating twice,

$$g(u, v) \sim u - K_0 \frac{u^3}{6} \quad \text{as } u \to 0$$

Thus for small u, the parameter v does not enter $g(u, v)$ until terms of order smaller than u^3.

Example 1:

To calculate the circumference of a geodesic circle of small radius r and to see how it differs from the Euclidean formula $2\pi r$.

In geodesic polar the circle is the parallel $u = r$. Hence $ds = g dv$ and the circumference C is

Hence

$$C = \int_0^{2\pi} g(r, v) dv \sim \int_0^{2\pi} \left(r - \frac{K_0}{6} r^3\right) dv = 2\pi \left(r - \frac{K_0}{6} r^3\right)$$

to the first significant term, where K_0 is the Gaussian curvature at the centre of the circle.

This suggests another intrinsic formula for K. Let C be the circumference of the geodesic circle of centre P and radius r. Then

$$(K)_P = \lim_{r\to 0} \frac{2\pi r - C}{\frac{1}{3}\pi r^3}$$

Exercises:

1.Prove that, if A is the area of a geodesic disk of centre P and radius r, then

$$(K)_P = \lim_{r\to 0} \frac{\pi r^2 - A}{\frac{1}{12}\pi r^4}$$

3.9. Surfaces of constant curvature

If K has the same value K_0 at every point of a surface, the surface is said to have constant curvature K_0.

Minding's theorem. Two surfaces of the same constant curvature are locally isometric. Strictly, if P is any point of one of these surfaces and $\bar{P}$ is any point of the other, then $\bar{P}$ has a neighbourhood which is isometric with a neighbourhood of P, the points P and $\bar{P}$ being corresponding points. In what follows, 'surface' means a sufficiently small region.

We prove this theorem by showing that if S is a surface with constant curvature K_0, then
(1) if $K_0 = 0, S$ is isometric with a plane;
(2) if $K_0 = 1/a^2, S$ is isometric with a sphere of radius a; and
(3) if $K_0 = -1/a^2, S$ is isometric with a certain surface of revolution, called a pseudo-spkere, determined by the value of a.
In each case a given point of S can be mapped into a prescribed point of the plane, sphere, or pseudo-sphere.

The theorem for two surfaces S and $\bar{S}$ with the same K then follows by mapping each surface isometrically on to the same plane, or sphere, or surface of revolution, so that given points P and $\bar{P}$ correspond to the same point.
Let P be a given point of the surface S of constant curvature K_0, and let C be a geodesic through P. Take as parametric curves the geodesics orthogonal to C together with their

orthogonal trajectories. Let $v = c$ be the geodesic orthogonal to C at a point distance c from P measured along C, and let $u = c$ be the parallel orthogonal to the curves $v =$ constant and at a distance c from the parallel C measured along the geodesic. Then u, v is a parameter system in the neighbourhood of P, and the metric of the surface is of the form $du^2 + g^2 dv^2$

for some $g(u, v)$. Since $u = 0$ is the geodesic C, it follows from (10.8) that $\partial g^2/\partial u = 0$ when $u = 0$. Also, v is the arcual distance along C, i.e. $ds = dv$ when $u = 0$, so that $g = 1$ when $u = 0$. Hence, $(g)_{u=0} = 1, (g_1)_{u=0} = 0$ ....(1)

Using now the formula $K = -g_{11}/g$ proved in section 17, $g(u, v)$ satisfies the partial differential equation $g_{11} + K_0 g = 0$ ………..(2)

with boundary conditions (1) these are sufficient to determine g when K_0 is given.

Case (1), $K_0 = 0$

When $g_{11} = 0, g_1$ is a function of v only and therefore $g_1 = 0$ since $(g_1)_{u=0} = 0$. From $g_1 = 0$ it follows that g is a function of v only and is therefore 1 since $(g)_{u=0} = 1$. With $g = 1$, the metric is $du^2 + dv^2$

i.e. the metric of a plane when u, v are taken as Cartesian coordinates. Hence, the surface S in the neighbourhood of P is isometric with a region of the plane.

This confirms that K is a satisfactory measure of curvature for a surface since its vanishing is both necessary and sufficient for the surface to be isometric with a plane.

Case (2), $K_0 = 1/a^2$

Equation (2) integrates to give

$$g(u, v) = A(v)\sin\frac{u}{a} + B(v)\cos\frac{u}{a}$$

Hence $(g_1)_{u=0} = (1/a)A(v) = 0$, and $(g)_{u=0} = B(v) = 1$, so that $g = \cos(u/a)$ and the metric is

$$du^2 + \cos^2\frac{u}{a} dv^2$$

This is the metric of a sphere of radius a. (The more usual metric is given by the transformation

$u = a\left(\frac{1}{2}\pi - \bar{u}\right), v = a\bar{v}$.) The surface S in the neighbourhood of P is therefore isometric with a region of a sphere of radius a.

Case (3), $K_0 = -1/a^2$

By arguments similar to those for case (2), $g = \cosh(u/a)$ and the metric of S in the neighbourhood of P is

$$du^2 + \cosh^2\frac{u}{a}dv^2$$

This form, in which E, F, G are functions of u only, shows that S is isometric with a certain surface of revolution (cf. Exercise 8.3).

Writing $u = a\bar{u}, v = a\bar{v}$, the metric becomes

$$a^2(d\bar{u}^2 + \cosh^2\bar{u}d\bar{v}^2)$$

This is the metric of the surface obtained by revolving the curve

$$x = a\cosh\bar{u}, y = 0, z = a\int_0^{\bar{u}}\sqrt{(1 - \sinh^2\theta)d\theta}\ (|\bar{u}| < \log(1 + \sqrt{2}))$$

about the z-axis.

This completes the proof of the theorem on the isometries of surfaces of constant curvature. The metrics and surfaces constructed above are special, chosen to prove the theorem as simply as possible. There are, however, other surfaces of revolution with constant curvature, since any function $g(u)$ which satisfies (2) (but not the boundary conditions (1)) gives a metric which can be transformed into the standard metric of a surface of revolution. For example, when $K_0 = -1/a^2$, g can be taken to be $ae^{u/a}$. Writing $u = a\bar{u}$, the metric becomes $a^2(d\bar{u}^2 + \rho^{2\bar{u}}dv^2)$,

which is therefore the metric of a surface of constant curvature $-1/a^2$. An important example of a surface of constant zero curvature is the surface generated by the tangents to any space curve. If $\mathbf{r}(s)$ is the position vector of a point on the curve, in terms of the are s as parameter, then a point on the surface is given by $\mathbf{r}(s) + v\mathbf{t}(s)$ where s and v are the parameters. The fundamental coefficients are

$E = 1 + \kappa^2v^2, F = 1, G = 1, H = |\kappa v|$where κ, the curvature of the curve, is a function of s only.

UNIT IV

Non Intrinsic properties of a surface: The second fundamental form- Principal curvature – Lines of curvature – Developable - Developable associated with space curves and with curves on surface - Minimal surfaces – Ruled surfaces.

Chapter 4: Sections 4.1 to 4.8.

4.1. The Second Fundamental Form:

Theorem 1:

Let $\vec{r} = \bar{r}(u, v)$ be the eqn of the Surface curvature vector then

$\bar{r}'' = k_n \bar{N} + \lambda \bar{r}_1 + \mu r_2 \quad \ldots\ldots(1)$

where, $k_n =$ normal curvature in the normal component of $\bar{r}''$

$\bar{N} =$Unit vector normal to the surface.

and $\lambda \bar{r}_1 + \mu \bar{r}_2 =$ The vector with components.

∴ Taking (.) product to equation (1) with $\bar{N}$.

$\therefore (1) \Rightarrow \bar{r}'' \cdot \bar{N} = k_n \vec{N} \cdot N + \lambda \overline{r_1} \cdot \bar{N} + \mu \overline{r_2} \cdot \bar{N}.$

$\Rightarrow \bar{N} \cdot \bar{r}'' = k_n \ldots\ldots\ldots\ldots (2)$

$[\because \bar{N}.\bar{N} = 1$ and $\bar{r}_1, \bar{r}_2$ lies on the tangent plane]

$\bar{N} \cdot \bar{r}_1 = 0, and\ \bar{N} \cdot \bar{r}_2 = 0$

Also we know that,

$$\bar{r}' = \frac{d}{ds}(\bar{r})$$
$$= \frac{\partial}{\partial u}(\bar{r}) \cdot \frac{du}{ds} + \frac{\partial}{\partial v}(\bar{r}) \frac{dv}{ds}$$

$\bar{r}' = \bar{r}_1 u' + \bar{r}_2 u' \quad \ldots\ldots\ldots..(3)$

$$\bar{r}'' = \frac{d}{ds}[\bar{r}_1 u' + \bar{r}_2 v'].$$
$$= \bar{r}_1 u'' + \bar{r}_2 v'' + u'\left[\frac{d}{ds}(\bar{r}_1)\right] + v'\left[\frac{d}{ds}(\bar{r}_2)\right]$$
$$= \bar{r}_1 u'' + \overline{r_2} v'' + u'\left[\frac{\partial}{\partial u}(\bar{r}_1)\frac{du}{ds} + \frac{\partial}{\partial v}(\bar{r}_1)\frac{dv}{ds}\right]$$
$$+ v'\left[\frac{\partial}{\partial u}(\bar{r}_2)\frac{du}{ds} + \frac{\partial}{\partial v}(\bar{r}_2) \cdot \frac{dv}{ds}\right]$$
$$= \bar{r}_1 u'' + \overline{r_2} v'' + u'[\bar{r}_{11} u' + \bar{r}_{12} v'] + v'[\bar{r}_{21} u' + \bar{r}_{22} v']$$
$$\bar{r}'' = \bar{r}_1 u'' + \bar{r}_2 v'' + \bar{r}_{11} u'^2 + \bar{r}_{12} u'v' + \bar{r}_{21} u'v' + \bar{r}_{22} v_1'^2] \ldots\ldots..(4)$$

Taking (.) product to (4) with $\vec{N}$,

(4) $\cdot \bar{N} \Rightarrow$

$\bar{r}'' \cdot \bar{N} = u^4\bar{r}_1 \cdot \bar{N} + v''\bar{r}_2 \cdot \bar{N} + u'^2 \cdot \bar{r}_{11} \cdot \bar{N} + u'v'\bar{r}_{12} \cdot \bar{N}$
$+u'v'\bar{r}_{21} \cdot \bar{N} + v'^2\bar{r}_{22} \cdot \bar{N}.$
$= 0 + 0 + u'^2\bar{r}_{11} \cdot \bar{N} + 2u'v'\bar{r}_{12} \cdot \bar{N} + v'^2 r_{22} \cdot \bar{N}$
$\bar{r}''. \bar{N} = Lu'^2 + 2Mu'v' + Nv'^2 [\bar{N} \cdot \bar{r}_1 = N \cdot \overline{r_2} = 0]$

where $L = \bar{r}_{11} \cdot \bar{N}, M = \bar{r}_2 \cdot \bar{N}$ & $N = r_{22} \cdot \bar{N}$

$$[by(2)]K_n = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

$$K_n = \frac{L(du)^2}{(ds)^2} + 2M\frac{(du)(dv)}{(ds)^2} + N\frac{(dv)^2}{(ds)^2}$$

$$K_n = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2} \dots\dots\dots (5)$$

Since, $(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$

∴ The Quadratic form $(du)^2 + 2Mdudu + N(dv$: is known as the second fundamental form.

where $L = \bar{N} \cdot r_{11}, M = \bar{N} \cdot \bar{r}_{12}, N = \bar{N} \cdot \bar{r}_{22}$ are the second fundamental coefficient.

Note:

1.Alternative Expressions For L, M, N: We know that, $\bar{N} \cdot \bar{r}_1 = 0$

Diff (6) w. r. to ' u ',

$\bar{N}_1 \cdot \bar{r}_1 + N \cdot \bar{r}_{11} = 0$
$\Rightarrow \bar{N} \cdot \bar{r}_1 = -\bar{N}_1\bar{r}_1$
$\Rightarrow L = -\bar{\bar{N}}_1 \cdot \bar{r}_1$

Diff (6) w.r.to 'v'

$N_2 \cdot \bar{r}_1 + N \cdot \bar{r}_{12} = 0$
$\Rightarrow \bar{N}_2 \cdot \bar{r}_{12} = -\bar{N}_2 \cdot \bar{r}_1$

Similarly, we know that $\bar{N} \cdot \bar{r}_2 = 0$

Diff (7) w. r. to ' u ' we get $M = -\bar{N}_1 \cdot \bar{r}_2$

Diff (7) w.r. to ' v ' we get $N = -\bar{N}_2 \cdot \bar{r}_2$

Thus, $L = -\bar{N}_1 \cdot \bar{r}_1, N = -\bar{N}_2 \cdot \bar{r}_2$ &

$M = -\bar{N}_2 \cdot \bar{r}_1 = -\bar{N}_1 \cdot \bar{r}_2$

2. $[\bar{r}_{11}, \bar{r}_1, \bar{r}_2] = \bar{r}_{11} \cdot (\bar{r}_1 \times \bar{r}_2)$

$= \bar{r}_{11} \cdot H\bar{N}$

$= H(\overline{r_{11}}, \bar{N})$

$= HL$

$\Rightarrow L = \frac{1}{H}[\bar{r}_{11}, \bar{r}_1, \bar{r}_2]$

$M = \frac{1}{H}[\bar{r}_{12}, \bar{r}_1, \bar{r}_2]$

$\& \ N = \frac{1}{H}[\bar{r}_{22}, \bar{r}_1, \bar{r}_2]$

Theorem 2: (Meusnier's Theorem)

If ' θ ' in the angle between the Surface normal ' $\bar{N}$ ' & the principal normal ' $\bar{n}$ ' then $k_n = k\cos\theta$.

Proof:

Since ' θ ' in the angle be tween $\bar{N}$ & $\bar{n}$.

$\therefore \bar{N} \cdot \bar{n} = |\bar{N}||\bar{n}|\cos\theta \rightarrow (A) \left[\because \cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right]$

[But we know that, $\bar{r}'' \cdot \bar{N} = k_n$

$\therefore$ Again, $\bar{r}'' = \frac{d}{ds}(\bar{r}') = \frac{d}{ds}(\bar{t}) = \bar{t}' = \kappa\bar{n}$

$\Rightarrow \bar{r}'' = k\bar{n}$ sub in (1)

$\therefore k' \cdot \bar{N} = k_n$

$\Rightarrow k(|\bar{N}||\bar{n}|\cos\theta) = k_n$

$\Rightarrow K'(\cos\theta) = k_n]$

Taking (.) with $\bar{N}$

$\Rightarrow k\bar{n}\cdot\bar{N} = k_n\bar{N}\cdot\bar{N} + \lambda\bar{r}_1\cdot\bar{N} + \mu\bar{r}_2\cdot\bar{N}$

$k|\bar{N}||\bar{n}|\cos\theta = k_n + 0 + 0.$

[by eqn $(A), \bar{N}\cdot\bar{N} = 1, \bar{r}_1\cdot\bar{N} = \bar{r}_2\cdot\bar{N} = 0]$

$\Rightarrow k\cos\theta = k_n$ [: $|\bar{N}||\bar{n}| = 1$].

Normal curvature:

Let ' p ' be the point surface $\bar{r} = \bar{r}(u, v)$ consider a curve $r = r(s)$ through the point $\bar{r}$ " along the normal to the surface is defined to be normal curvature of the curve at the point. P.

Thus $k_n = \bar{r}''\cdot\bar{N}$

Classification of point

We know that, the normal curvature is given by

$K_n = \frac{L(du)^2 + 2Mdudv + N(dv)^2)}{E(du)^2 + 2Fdudv + G(dv)^2}$ ……….. (1)

$\therefore$ The denominator of the equation (1) R.H.S is positive definite.

So the sign of K_n depends only upon the sign of the Numerator.

$\therefore Ldu^2 + 2Mdudv + Ndv^2 = \frac{1}{L}[L^2du^2 + 2MLdudv$

$+NLdv^2]$

$= \frac{1}{L}[[Ldu + Mdv]^2 - M^2dv^2 + NLdv^2]$

$= \frac{1}{L}[[Ldu + Mdv]^2 + (LN - M^2)dv^2]$

Here $L > 0$. So sign of k_n depends on the sign of $LN - M^2$

1) If $LN - M^2 > 0$

[(i.e) if at a pt ' p ' on the surface this form is definite] then ' P ' is called an elliptic point.

[(i.e.) then k_n maintains the same sign for all derivate directions at ' P '].

2) If $LN - M^2 = 0$ then ' P ' is called a parabolic point

[(i.e) κ_n retains the same sign for all directions through' ' p ' except one for which the curvature is zero l.

3) If $LN - M^2 < 0$ then ' P ' is called a Hyperbdic point

$$\text{(i.e.) } K_n = \begin{cases} \text{Positive for directions lying with in a certain angle} \\ \text{Negative for directions lying outside this angle} \\ 0 \qquad\qquad \text{for along the directions which form the angle} \end{cases}$$

and the critical directions are called the Asymptotic directions.

Theorem 3:

A Geometrical Interpretation of the second fundamental Form.

Let $P(u, v)$ & $Q(u + h, v + k)$ be near points on a Surface and let ' d ' be the perpendicular distance from a onto the tangent plane to the surface at P.

If r_P and r_Q are the position vectors of P & Q then

$d = \frac{1}{2}[Lh^2 + 2mhk + Nk^2] + O(h^3, k^3)$.

Proof:

i) $P(u, v)$ & $Q(u + h, v + k)$ be two near points on the Surface

ii) $d =$ The perpendicular distance from ' Q ' onto the tangent plane to the surface at ' P '.

If $r_P =$ The position vector of P

$\gamma_Q =$ The position vector of Q

To prove that d $= \frac{1}{2}[Lh^2 + 2Mhk + Nk^2] + O(h^3, k^3)$

We know that,

$$f(\bar{b})-f(\bar{a})==\sum_{k=1}^{m-1}\frac{1}{k!}f^{k}(\bar{a};b-a)$$

$$+\frac{1}{m!}f^{m}(z;\bar{z}-\bar{a}) \qquad \ldots..(1)$$

$$k=1, f^{(1)}(\bar{x};\bar{t})=\sum_{i=1}^{n} D_i f(x) \ldots\ldots.(2)$$

$$k=2, f^{(2)}(\bar{x},\bar{t})=\sum_{i}^{n}\sum_{j}^{n} D_{ij} f(\bar{x}) t_i \text{t} \ldots\ldots\ldots.(3)$$

Taking, $\bar{b}=\bar{r}+d\bar{r}$ & $\bar{a}=\bar{r}$ in (1)

$\therefore$ (1) $\Rightarrow$

$$f(\bar{r}+d\bar{r})-f(\bar{r})=\frac{1}{1!}f'(\bar{r};d\bar{r})+\frac{1}{2!}f^{(2)}(\bar{r};d\bar{r})$$

$$\Rightarrow(\bar{r}+d\bar{r})-\bar{r}=[D_1(\bar{r})du+D_2(\bar{r})du]+\frac{1}{2}[D_{1,1}(\bar{r})dudu+P_{1,2}(\bar{r})dudv]$$

$$+D_{2,1}(\bar{r})dvdu+D_{2,2}\bar{\text{r}}dvdv$$

$$=\bar{r}_1du+\bar{r}_2dv+\frac{1}{2}[\bar{r}_{11}du^2+2\bar{r}_{12}dudv+\bar{r}_{22}du^2] \ldots\ldots\ldots\ldots(4)$$

Let $r_P=$ the position vector of $P=\bar{\gamma}$

& $\gamma_Q=$ the position vector of $Q=\bar{r}+d\bar{r}$

then, $d=$ projection of $\overline{PQ}$ on $\bar{N}$

$$=\overline{PQ}\cdot\bar{N}$$

$$=(\overline{OQ}-\overline{OP})\cdot\bar{N}.$$

$$=[(\bar{r}+d\bar{r})-\bar{r}]\cdot\bar{N}$$

$$=\left[\bar{r}_1du+\bar{r}_2dv+\frac{1}{2}[\bar{r}_{11}du^2+2\bar{r}_{12}dudv+\bar{r}_{22}dv^2]\right]\frac{1}{N}$$

using the transformation and negative higher power of second order differential.

$$\therefore d=\left[\bar{r}_1\cdot\bar{N}du+\bar{r}_2\cdot\bar{N}du+\frac{1}{2}[\bar{r}_{11}\cdot\bar{v}du^2+2\bar{r}_{12}\cdot\bar{N}dudv+\bar{r}_{22}Ndv^2]\right]$$

$$=0+0+\frac{1}{2}[Ldu^2+2Mdudv+Ndv^2]$$

$$d=\frac{1}{2}[Ldu^2+2Mdudv+vdv^2]$$

$$\Rightarrow 2d=Ldu^2+2mdudv+vdv^2$$

Thus, $2\times$ [length of the perpendicular from Q to the tangent plane at P] $=$ the second fundamental form.

Note:

1. At an elliptic point 'd 'retains the same sign.

   $\Rightarrow$ The surface wear p lies entirely on one side of the tangent plane at p.
2. At a hyperbolic point the surface crosses over the tangent plane.
3. Any point on an ellipsoidal surface is elliptic
4. Any point on a circular cylinder is parabolic.
5. Any point on the hyperbolic paraboloid $[x = u, y = v, z = u^2 - v^2]$ is hyperbolic

Example 1:

For a helicoid every point is a hyperbolic point.

Proof:

We know that equation of the helicoids is

$\bar{\gamma} = (u\cos v, u\sin v, av)$ ....(1)

Diff (1) w.r.to 'u' $\therefore \bar{r}_1 = (\cos v, \sin v, 0)$ ... ... (2)

Diff (1) w.r.to ' v '

$\therefore \bar{r}_2 = (-u\sin v, u\cos v, a)$ .... (3)

Diff (2) w.r.to ' u ' (2) $\Rightarrow r_{11} = (0,0,0)$ ... ... (4)

Diff (2) w.r.to 'v' (2) $\Rightarrow \bar{r}_{12} = (-\sin v, \cos v, 0) = \overline{r_{21}}$ ... ... (5)

Diff (3) w.r.to ' v ', (3)$\bar{r}_{22} = (-u\cos v, -u\sin v, 0)$ ... ... (6)

$\therefore$ From (2), (3) & (4) we get,

$$HL = [\bar{r}_{11}, \bar{r}_1, \bar{r}_2] = \begin{vmatrix} 0 & 0 & 0 \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & a \end{vmatrix}$$

$$HL = 0 \Rightarrow L = 0 \Rightarrow M = \frac{-a}{H}$$

Similarly, from (2), (3) & (5) we get, $HM = [\bar{r}_{12}, \bar{r}_1, \bar{r}_2] = -a$

and from (2), (3) & (6) we get $HN = [\bar{r}_{22}, \bar{r}_1, \bar{r}_2] = 0$

$$\therefore LN - M^2 = (0)(0) - \left(\frac{-a}{H}\right)^2$$

$$LN - M^2 = -\frac{a^2}{H^2} < 0$$

Thus all the point of a helicoid is hyperbolic.

$\bar{r} = (u\cos v, u\sin v, u^2)$ are elliptic.

4.2.Principal curvatures:

Section of the surface:

A plane drawn through a pt on a surface, cuts It in a curve, called the section of the surface.

Normal section & Oblique section:

If the plane is so drawn that it contains the normal to the surface then the curve is Called normal section, otherwise the section is called oblique Section.

Normal curvature (**Alternative Definition of Normal curvature**)

Let p be a pt on a surface $\bar{r} = r(u, v)$ the normal curvature at p in the direction (du, dv) is defined to be the curvature at ' p ' of the normal section parallel to the direction (du, dv)

Principal section, Principal curvature Mean curvature & Principal Direction:

The normal sections of a Surface which have greatest and least curvature are called "principal Sections".

The maximum and minimum curvature are called "principal curvature"and denoted by 'κ_a' and 'κ_b' .

Mean curvature $= \frac{K_a+K_b}{2} = \mu$

The direction of the principal section are called the principal direction & they are mutually orthogonal.

Gaussian curvature:

Gaussian curvature $= K = K_a \cdot K_b$.

The normal curvature at ' p ' in a direction (K_n now denote post K)

Specified by direction coefficients (l, m) is given by

$k = Ll^2 + 2mlm + Nm^2$ …… (1)

Where, $El^2 + 2Flm + Gm^2 = 1$ ….(2)

$\left[\because, k_n = \frac{Lu^2+2Muv+Nv^2}{Eu^2+2Fuv+Gv^2}\right]$ #

As l, m Vary, Subject to $El^2 + 2Flm + Gm^2 = 1$, the normal curvature will vary.

To find the extreme values: [using Lagrange's Multiplies].

$K = Ll^2 + 2Mlm + Nm^2 - \lambda[El^2 + 2Flm + Gm^2 - 1]$ … … … . (3)

when K is stationary,

Diff (3) w.r.to ' l ', (partially)

$$\therefore \frac{\partial k}{\partial l} = 2Ll + 2mm - \lambda[2El + 2Fm] = 0$$

Divided by 2, $\frac{1}{2}\frac{\partial k}{\partial l} = Ll + mM - \lambda[El + Fm] = 0$ …………(4)

Diff (3) w.r.to ' m ' (partially) $\frac{\partial k}{\partial m} = 2ml + 2Nm - \lambda[2Fl + 2Gm] = 0,$

Divided by 2, $\frac{1}{2}\frac{\partial k}{\partial m} = ml + Nm - \lambda[Fl + Gm] = 0,$ ………..(5)

$$((4) \times l) + (5) \times m) \Rightarrow$$
$$Ll^2 + Mml - \lambda[El^2 + Fml] + [mlm + Nm^2 - \lambda[Flm + Gm^2] = 0$$
$$\Rightarrow Ll^2 + Mml - \lambda El^2 - \lambda Fml + mlm + Nm^2 - \lambda Flm - \lambda Gm^2 = 0$$

$$\Rightarrow Ll^2 + 2mml + Nm^2 - \lambda[El^2 + 2Fml + Gm^2] = 0$$
$$\Rightarrow K - \lambda(1) = 0 \ [\because (1) \ \& \ (2)]$$
$$\Rightarrow K = \lambda$$

Thus the extreme values of k are obtained when $\lambda = k$

To find principal curvature from Quadratic equation by elimination of l, m from (4) & (5):

eliminate l & m.

$$(4) \Rightarrow Ll + mM - \lambda El - \lambda Fm = 0$$
$$\Rightarrow l[L - \lambda E] + m[M - \lambda F] = 0$$
$$\Rightarrow l[L - \lambda E] = -m[M - \lambda F]$$
$$\Rightarrow \frac{l}{m} = \frac{-[M - \lambda F]}{[L - \lambda E]} \quad \dots\dots\dots..(6)$$
$$(5) \Rightarrow ml + Nm - \lambda[Fl + Gm] = 0$$
$$\Rightarrow Ml + Nm - \lambda Fl - \lambda Gm = 0$$

$$\Rightarrow l[m - \lambda F] + m[N - G\lambda] = 0$$
$$\Rightarrow l[M - \lambda F] = -m[N - G\lambda]$$
$$\Rightarrow \frac{l}{m} = \frac{-[N - G\lambda]}{[M - \lambda F]} \quad \dots\dots\dots\dots(7)$$

From (6) & (7) L.H.S are equal ⇒ R.H.S also equal

$$\frac{[M - \lambda F]}{[L - \lambda E]} = \frac{[N - G\lambda]}{[M - \lambda F]}$$
$$\Rightarrow [M - KF][M - KF] = [N - G \neq K][L - KE][\because \lambda = K]$$
$$\Rightarrow M^2 - 2KFM + K^2F^2 = NL - KEN - KGL + K^2GE$$
$$\Rightarrow M^2 - 2KFM + K^2F^2 - NL + KEN + K^{GL} - K - K^2GE = 0$$
$$\Rightarrow -K^2[EG - F^2] + K[EN + GL - 2FM] - (LN - M^2) = 0.$$

Multiply by (-1), ⇒ $K^2[EG - F^2] - K[EN + GL - 2FM] + (LN - M^2) = 0$ ….…….(8)

This is the quadratic equation in k.

∴ It gives two values for κ which correspondence to the extreme value of k.

This two values of k are denoted by k_a & k_b. Where k_a & k_b are the principal curvature at p

To find mean curvature:

We know that, Mean curvature $= \mu = \frac{1}{2}[k_a + k_b]$

[Also we know that, If $ax^2 + bx + c = 0 \& \alpha, \beta$ are two roots then

Sum of the roots $= \alpha + \beta = -b/a$

and the product of the roots $= \alpha\beta = c/a$]

$$\therefore \text{Mean curvature } = \mu \quad = \frac{1}{2}[K_a + K_b]$$
$$= \frac{1}{2}\left[-\frac{\text{co-eff of } K}{\text{co-eff of } K^2}\right]$$

$$\text{Mean curvature } = \frac{1}{2}\left[\frac{EN + GL - 2FM}{EG - F^2}\right]$$

To find Gaussian curvature:

We know that,

$$\text{Gaussian curvature } = \kappa = K_a \cdot K_b$$
$$= \left[\frac{\text{constant}}{\text{co} - \text{eff of } K^2}\right]$$
$$\text{Gaussian curvature } = K = \frac{EN - M^2}{EG - F^2}$$

To find the principal direction by eliminate λ :

(or) Elimination of λ from (4) and (5).

(4) $\Rightarrow Ll + mM = \lambda[El + Fm]$

$$\lambda = \frac{Ll + mM}{[El + Fm]} \dots \dots (9)$$

(5) $\Rightarrow \text{Ml} + Nm = \lambda[Fl + Gm]$

$$\lambda = \frac{\text{Ml} + Nm}{Fl + Gm} \quad \dots\dots\dots\dots (10)$$

From (9) & (10) $\Rightarrow$ LHS are equal $\Rightarrow$ R.H S are equal.

$$\frac{[L\ell + mM]}{[E\ell + FM]} = \frac{[M\ell + Nm]}{[F\ell + Gm]}$$

$$\Rightarrow [L\ell + mM][Fl + Gm] = [Ml + Nm][El + Fm]$$
$$\Rightarrow LFl^2 + GLlm + MF\text{ml} + GMm^2 - MEl^2 - MF\ell m - NElm - NFm^2 = 0.$$

$$\Rightarrow l^2[LF - ME] + lm[GL - NE] + m^2[GM - NF] = 0$$

$$\Rightarrow (EM - LF)\left(\frac{l}{m}\right)^2 + (EN - LG)\left(\frac{l}{m}\right) + (FN - GM) = 0. \dots \dots \dots (11)$$

∴ The roots of this eqn-gives the direction of the principal direction.

The discriminant of this equation $= (EN - LG)^2 - 4(EM - LF)(FN - GM) \dots (12)$

Consider, $FN - GM = FN \cdot \frac{E}{E} - GM \cdot \frac{E}{E} - \frac{FGL}{E} + \frac{FGL}{E}$

$\Rightarrow FN - GM = \frac{F}{E}[EN - GL] - \frac{G}{E}[-FL + EM]$ sub in (12).

(12) ⇒ The discriminant of eqn (II)

$$=(EN - GL)^2 - 4(EM - FL)\left[\frac{E}{E}(EN - GL) - \frac{G}{E}(-FL + EM)\right]$$

$$=(EN - GL)^2 - 2(EN - GL) \cdot \frac{2(EM - FL)F}{E} + \frac{4G}{E}(EM - FL)^2$$

$$+\frac{4F^2}{E^2}(EM - FL)^2 - \frac{4F^2}{E^2}(EM - FL)^2$$

$$=(EN - GL) - \frac{2E}{E}(EM - FL)J^2 + \frac{4(EM - FL)^2}{E^2}[-F^2 + EG] \ldots\ldots (13)$$

If the R.H.S of equation$(13) > 0$, then the equation (11) has distinct real roots and two distinct principal directions.

If the R.H.S of equation (13) $= 0$. Then, the roots will be consider.

when, $EN - GL = EN - FL = EM - FL = 0$.

(ie) when $\frac{E}{L} = \frac{G}{N} = \frac{F}{m}$ ……..(14)

Suppose, equation (14) is true at a point.

(i.e.) when L, M, N are proportional to E, F, G then The principal directions are indeterminate and the normal curvature is the same in all the directions

To prove that the principal directions are orthogonal.

(i.e.) To prove that the angle ' θ ' between the principal direction $= \pi/2$.

(i.e.) T.P.T: $\theta = \pi/2$.

From (11) ⇒ The principal directions are given by.

$$(EM - FL)^2\left(\frac{l}{m}\right)^2 + (EN - GL)\left(\frac{l}{m}\right) + (FN - GM) = 0$$

Let the roots of this equation. be $\frac{l}{m}$ *and* $\frac{l'}{m'}$

$$\left.\begin{aligned} \therefore \frac{l}{m} + \frac{l'}{m'} &= -\frac{(EN - GL)}{(EM - FL)} \\ \& \frac{l}{m} \cdot \frac{l'}{m'} &= \frac{(FN - GM)}{(EM - FL)} \end{aligned}\right\} \ldots\ldots\ldots (15)$$

∴ The angle ' θ ' both the principal direction is,

$\cos\theta = Ell' + F(lm' + ml') + G(mm')$

$$\therefore \cos\theta = mm'\left[E\frac{l}{m}\cdot\frac{l'}{m'} + F\left[\frac{l}{m}+\frac{l'}{m'}\right] + G\right]$$

$$= m'\left[E\left[\frac{FN-GM}{EM-FL}\right] + F\left[\frac{-EN+GL}{EM-FL}\right] + G\right] \quad [\because (15)]$$

$$= \frac{mm'}{FM-FL}[EFN - EGM + FEN + FGL + GEM - GFL]$$

$\cos\theta = 0$

$\Rightarrow \theta = \cos^{-1} 0$

$\Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ The principal directions are orthogonal.

Umbilic:

A point at which $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ is called an umbilic

Example:

On a Sphere every point is umbilic.

$\bar{r} = (a\sin u\cos v, a\sin u\sin v, a\cos u)$

4.3. Lines of curvature:

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature. (or)

A line of curvature on any Surface is a curve Such that the tangent at any point is a tangent line to the principal sections of the Surface at the point.

Theorem 1:

Rodrigues Theorem:

The necessary and sufficient condition for a curve to be a line curvature is that

$Kdr + d\bar{N} = 0$ on a surface

Proof:

Necessary part:

Assume that a curve on a Surface be a line of curvature.

$\therefore$ The tangent to this curve, at each pt of the curve is along a principal direction

(i.e.) The direction (du, dv) of this curve at each point (u,v) is along the principal direction.

To prove that $Kd\bar{r} + d\bar{N} = 0$

We know that, the principal directions are on by,

$[L - KE]du + [M - KF]dv = 0 \quad \ldots\ldots\ldots.(1)$

$[M - KF]du + [N - KG]dv = 0 \quad \ldots\ldots\ldots(2)$

where $k =$ principal curvature.

Also we know that,

$E = \bar{r}_1 \cdot \bar{r}_1, F = \bar{r}_1 \cdot \bar{r}_2, G = \bar{r}_2 \cdot \bar{r}_2, L = -\bar{N}_1 \cdot \bar{r}_1$………. (3)

Sub (3) in (1) and (2)

$\therefore$ (1) $\Rightarrow [-\bar{N}_1\bar{r}_1 - k(\bar{r}_1, \bar{r}_1)]du + [-\overline{N_2} - \bar{r}_1 - k(\bar{r}_1 \cdot \bar{r}_2)]dv = 0$

& (2) $\Rightarrow [-\bar{N}_1 \cdot \bar{r}_2 - k(\bar{r}_1 \cdot \bar{r}_2)]du + [-\bar{N}_2 \cdot \bar{r}_2 - k(\bar{r}_2 \cdot \bar{r}_2)]dv = 0.$

(i.e.) (1) $\Rightarrow -\overline{N_1}\bar{r}_1 du - k\bar{r}_1 \cdot r_1 du - \overline{N_2} \cdot \overline{r_1} du - k(\overline{r_1} \cdot \bar{r}_2)dv = 0$

& (2) $- \overline{N_1} \cdot \overline{r_2} du - k\bar{r}_1 \cdot \bar{r}_2 du - \bar{N}_2 \cdot \bar{r}_2 dv - k \cdot \overline{r_2} \cdot \overline{r_2} dv = 0$

(i.e.) $(-1) \times (1) \Rightarrow \bar{N}_1\bar{r}_1 du + \bar{N}_2\bar{r}_1 dv + h\bar{r}_1 \cdot \bar{r}_1 du + k_1\bar{r}_2 \cdot \bar{r}_2 dv = 0$

$\&(-1)(2) \Rightarrow \bar{N}_1 \cdot \bar{r}_2 du + \bar{N}_2 \cdot \bar{r}_2 dv + k\bar{r}_1 \cdot \bar{r}_2 du + k\bar{r}_2 \cdot \bar{r}_2 du = 0$

(1) $\Rightarrow (\bar{N}_1 du + \bar{n}_2 du) \cdot \bar{r}_1 + k[\bar{r}_1 du + \bar{r}_2 du]\bar{r}_1 = 0.$

& (2) $\Rightarrow (\bar{N}_1 du + \bar{N}_2 dv) \cdot \bar{r}_2 + k[\bar{r}_1 du + \bar{r}_2 du]\bar{r}_2 = 0.$

(1) $\Rightarrow \left[\frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} dv\right] \cdot \bar{r}_1 + k\left[\frac{\partial \bar{\gamma}}{\partial u} du + \frac{\partial \bar{r}}{\partial v} dv\right]\bar{r}_1 = 0.$

& (2) $\Rightarrow \left[\frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} du\right] \cdot \bar{\gamma}_2 + K\left[\frac{\partial \bar{\gamma}}{\partial u} du + \frac{\partial \bar{\gamma}}{\partial v} du\right]\bar{\gamma}_2 = 0$

(1) $\Rightarrow dN \cdot \bar{r}_1 + kd\bar{r} \cdot \bar{r}_1 = 0$ &

(2) $\Rightarrow dN \cdot r_2 + Kdr \cdot r_2 = 0.$

(ie) $[d\bar{N} + kd\bar{r}] \cdot \bar{r}_1 = 0$ ………..(4)

& $[d\bar{N} + kd\bar{r}] \cdot \bar{r}_2 = 0.$ ……….. (5)

From equation (4) & (5) we get,

$d\bar{N} + kd\bar{r}$ is perpendiculur to both $\bar{r}_1$ and $\bar{r}_2$ ……….(6)

But $\bar{N}^2 = \bar{N} \cdot \bar{N} = 1$

$\Rightarrow d\bar{N} \cdot \bar{N} + \bar{N} \cdot d\bar{N} = 0$
$\Rightarrow 2\bar{N} \cdot d\bar{N} = 0$
$\Rightarrow \bar{N} \cdot d\bar{N} = 0$

$\Rightarrow d\bar{N}$ is orthogonal to $\bar{N}$

$\Rightarrow d\bar{N}$ is a tangent vector ………..(7)

Further, $d\bar{r} = \bar{r}_1 du + \bar{r}_2 du \& \bar{r}_1, r_2$ are tangential vectors

$\Rightarrow \kappa d\bar{r}$ is also a tangential vector.

From (7) & (8)

The vector $d\bar{N} + kd\bar{r}$ is a tangential vector

(i.e) $d\bar{N} + Kd\bar{r}$ is a vector on the tangent plane.

If $d\bar{N} + kdr \neq 0$ then from (4) & (5) $d\bar{N} + kd\bar{r}$ is parallel to $\bar{N}$.

This is contradiction to (9)

$\therefore\ d\bar{N} + Kd\bar{r} = 0$

Sufficient part:

Assume that $d\bar{N} + kd\bar{r} = 0$ ……….(10)

along a curve For any some Scalar function ' K '.

The along the curve we have

$(d\bar{N} + kd\bar{r}) \cdot \bar{r}_1 = 0 \cdot \bar{r}_1 = 0.$
$\&\ (d\bar{N} + kd\bar{r}) \cdot \bar{r}_2 = 0 \cdot \bar{r}_2 = 0$

$\therefore\ (du, dv)$ is the direction of the that cure at the pt (u, v) then by retracing the same Step as above, we see that, $\left.\begin{array}{l}(Ldu + Mdv) - K(Edu + Fdv) = 0 \\ \&(Mdu + Ndv) - K(Fdu + Gdv) = 0\end{array}\right\} \ldots\ldots..(11)$

Retracing the steps from (4) & (5) to (1) & (2) from (10)

$$d\bar{N} + kd\bar{r} = 0$$

$$\therefore \left[\frac{\partial \bar{N}}{\partial u} du + \frac{\partial \bar{N}}{\partial v} dv\right] + k\left[\frac{\partial \bar{r}}{\partial u} du + \frac{\partial \bar{r}}{\partial v} d\bar{v}\right] = 0$$

(i.e.) $\bar{N}_1 du + \bar{N}_2 dv + k(\bar{r}_1 du + \bar{r}_2 dv) = 0.$

$K(\bar{r}_1 du + \bar{r}_2 dv) = -\bar{N}_1 du - \bar{N}_2 dv$

post multiply by $\bar{r}_1 du + \bar{r}_2 dv$ on both sides

we get, $\kappa(\bar{r}_1 du + \bar{r}_2 dv)(\bar{r}_1 du + \bar{r}_2 dv) = (-\bar{N}_1 du - \overline{N_2} dv)(\overline{r_1} du + \overline{r_2} dv)$

$$\kappa[\bar{r}_1 \cdot \bar{r}_1 du^2 + 2\overline{r_1} \cdot \bar{r}_2 dudv + \bar{r}_2 \cdot \bar{r}_2 dv^2] = (-\overline{N_1} \cdot \bar{r}_1)du^2 + [-\overline{N_2} \cdot \overline{r_1} - \overline{N_1} \cdot \overline{r_2}]dudv + [-\overline{N_2}\overline{r_2}]dv^2$$

(i.e.) $\kappa[Edu^2 + 2Fdudv + Gdv^2] = Ldu^2 + 2Mdudv + Ndv^2$

[Since $E = \bar{r}_1^2, G_T = \bar{r}_2^2, F = \bar{r}_1.\bar{r}_2, L = -\bar{N}_1\bar{r}_1, M = -\bar{N}_2 - \bar{r}_1 \& N = -\bar{N}_2 \cdot \bar{r}_2$]

$$\therefore \kappa = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} = \kappa_n$$

$\Rightarrow k$ is a normal curvature at (u, v)

Hence, the direction (11) at any point of the curve gives the principal directions at (u, v).

The tangent to the curve at each point is along a principal direction at that point.

$\therefore$ The curve is a line of curvature

Minimal Surface:

Surface whose mean curvature is zero at all points are called minimal surface.

Theorem:

Lines of curvature as parametric curves

$F = 0, M = 0$ is the necessary and sufficient conditions for the lines of curvature to be parametric curves

(or)

If the parametric curves are lines of curvature the than $F = 0, M = 0$.

Proof:

Necessary part :

Let $\bar{r} = \bar{r}(u, v)$ be a gi. Surface then the differential eqn. of the lines of curvature is given by,

$[EM - FL](du)^2 + [EN - GL]dudu + [FN - GM](dv)^2 = 0 \ldots\ldots\ldots.(1)$

If the lines of curvature be takes as parametric curves then $F = 0$,

Since the principal directions are orthogonal $u =$ constant & $v =$ constant are the equations of parametric curves.

∴ combined differential equations of parametric curves is given by $dudv = 0$ ……….(2)

By hypothesis,

The lines of curvature coincide with the parametric curves at each point ………(3)

∴ (1) & (2) are identical or respect the same curves.

$\therefore EM - FL = 0 \ \ldots\ldots\ldots(4)$

$FN - GM = 0 \ldots\ldots\ldots..(5)$

$EM = 0$ and $GM = 0$ $[\because F = 0]$

$\Rightarrow M = 0$ $[\because E \neq 0 \;\&\; G \neq 0]$

Sufficient part:

If $M = 0$ & $F = 0$ then

the eqn (1) $\Rightarrow [EN - GL]dudv = 0$ but $EN - GL \neq 0$.

[IF $EN - GL = 0 \Rightarrow \frac{E}{L} = \frac{G}{N}$ which is the condition for umbilic point.]

Hence $dudv = 0$. which is the diff. equation of the parametric curves.

Theorem: Euler's Theorem

Let the lines of curvature be parametric curves then $k_n = k_a \cos^2\psi + k_b \sin^2\psi$

Where $k_n =$ The normal curvature at P along (du, dv)

$\kappa_a, \kappa_b =$ The principal curvatures.

$\psi =$ The angle between the direction (du, du) & the principal direction $dr = 0$. (or)

Let κ_a, κ_b be the principal curvature of a surface at any point p on it, then the normal curvature k_n at P in the direction making an angle ψ with the principal direction in which the normal curvature is k_n and is given by.

$k_n = k_a \cos^2 \psi + k_b \sin^2 \psi$

Proof:

Let the equation of the Surface $\bar{r} = \bar{r}(u, v)$,

If the lines of curvature be taken as parametric then $F = 0$ & $M = 0$.

& the normal curvature $\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$

Let ' p ' be any point on the surface.

Then the principal directions at ' P ' are the directions of the parametric curves

$v =$ constant & $u =$ constant.

We know that, the direction co-efficient in the direction of $V =$ constant at P is $\left(\frac{1}{\sqrt{E}}, 0\right)$

and the direction co-eff in the direction of $u =$ constant at p is $\left(0, \frac{1}{\sqrt{G}}\right)$

Let (l, m) be the direction co-eff of the direction in which the normal curvature in κ_n.

$$\therefore \cos\psi = Ell' + F[lm' + l'm] + Gmm'$$

$$= El\left(\frac{1}{\sqrt{E}}\right) + 0 + G(m)(0)$$

$$= \frac{El}{\sqrt{E}}$$

$$\Rightarrow \cos\psi = l\sqrt{E} \quad \ldots\ldots\ldots(1)$$

Since the principal directions cut at 90°, the angle between the direction in which the normal curvature is κ_a & the direction in which the normal curvature is κ_b

$$\therefore \cos(90 - \psi) = E(l)(0) + 0 + G(m)\left(\frac{1}{\sqrt{G}}\right)$$

We know that, the normal curvature K_n in the direction with direction co-eff (l, m) is given by,

$$K_n = \frac{1l^2 + 2Mlm + Nm^2}{El^2 + 2Flm + Gm^2}$$

$$K_n = Ll^2 + 2Mlm + Nm^2 \qquad [\because El^2 + 2Flm + Gm^2 = 1]$$

$$\Rightarrow K_n = Ll^2 + 0 + Nm^2 \quad [\because M = 0]$$

$$\Rightarrow K_n = Ll^2 + Nm^2 \quad \ldots\ldots\ldots\ldots(3)$$

$$\therefore K_a = L\left[\frac{1}{E}\right] + N(0)$$

$$k_a = \frac{L}{E} \quad \ldots\ldots\ldots\ldots(4)$$

Taking $l = 0$ & $m = \frac{1}{\sqrt{G}}$ in (1)

$$K_b = L(0)^2 + N\left(\frac{1}{\sqrt{G}}\right)^2$$

From (1), $l = \frac{\cos\psi}{\sqrt{E}}$

from (3), $m = \frac{\sin\psi}{\sqrt{G}}$

From (4) & (5)

$L = E\kappa_a$ & $N = G\kappa_b$

∴ From (3)

$$k_n = E\kappa_a\left[\frac{\cos\psi}{\sqrt{E}}\right]^2 + G\kappa_b\left[\frac{\sin\psi}{\sqrt{G}}\right]^2$$

$$\Rightarrow k_a = Ek_a\frac{\cos^2\psi}{E} + GK_b\frac{\sin^2\psi}{G}$$

$$\Rightarrow k_n = k_a\cos^2\psi + k_b\sin^2\psi$$

For the principal curvature on the Surface of revolution.

$\bar{r} = (u\cos\phi\,, u\sin\phi\,, f(u))$ is given by.

$$k_a = \frac{f_{11}}{(1+f_1^2)^{3/2}}, k_b = \frac{f_1}{u(1+f_1^2)^{1/2}}$$

Gaussian curvature $= k_1 = k_a \cdot k_b = \frac{f_1 f_{11}}{U(1+f_1{}^2)}$

Mean curvature (1st curvature)$= \frac{K_a+\dot{K}_b}{2}$

$$= \frac{1}{2}\left[\frac{uf_{11} + f_1(1+f_1)}{u(1+f_1^2)^{3/2}}\right]$$

Corollary:

The sum of the normal curvatures at any point on a surface in two directions of right angle is constant is equal to the sum of the principal curvatures at the point.

Proof:

Let κ_a & κ_b be the principal curvatures at any ' p ' on the given surface

Consider two directions at p which cut at $90°$.

Let κ_1, κ_2 be the normal curvature in these two direction. Let ψ_1 be the angle between the directions at P, in which the normal curvatures are k_a & k_1& ψ_2 be that of K_b & K_2

∴ By Euler's theorem,

$$k_1 = k_a\cos^2\psi_1 + k_b\cos^2\psi_1 \dots\dots\dots..(1)$$

[Replacing κ_n, ψ respectively by κ_1, ψ, in the Euler's formula]

$$k_2 = k_a \cos^2 \psi_2 + k_b \sin^2 \psi_2 [\because \psi_2 = 90 + \psi_1]$$
$$\Rightarrow k_2 = k_a \cos^2 \left(\frac{\pi}{2} + \psi_1\right) + k_b \sin^2 \left(\frac{\pi}{2} + \psi_1\right)$$
$$\Rightarrow k_2 = k_a \sin^2 \psi_1 + k_b \sin^2 \psi_1$$
$$(1) + (2) \Rightarrow \kappa_1 + \kappa_2 = \kappa_a [\cos^2 \psi_1 + \sin^2 \psi_1] + \kappa_b [\cos^2 \psi_1 + \sin^2 \psi_1]$$
$$= \kappa_a + \kappa_b.$$
$$\kappa_1 + \kappa_2 = \text{ constant .}$$

Thus the sum of the normal curvature at p. In any two directions which, at at $90°$ is the sum of the principal curvatures at P, a constant.

Elliptic, parabolic & Hyperbolic points:

Suppose k in the Gaussian curvature of a point $P(u, v)$ on a surface.

If κ_a & κ_b are principal curvatures at P, then

$$\kappa = \kappa_a \kappa_b = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{H^2}, \text{ where } H^2 = EG - F^2 > 0$$

Elliptic point:

If a point 'p' at a Gaussian curvature is positive (i.e.) If $LN - M^2 > 0$ then the pt ' p ' is called an elliptic point.

$$\left[K > 0 \Leftrightarrow \frac{LN - M^2}{H^2} > 0 \Leftrightarrow LN - M^2 > 0 \text{ since } H^2 > 0\right]$$

$\therefore$ A point is an elliptic point $\Leftrightarrow$ the principal curvatures $(\kappa_a$ & $\kappa_b)$ at the point are of the same signs.

CA Pt, P is an elliptic point.

$\Leftrightarrow k > 0 \Leftrightarrow k_a k_b > 0 \Leftrightarrow$ both k_a & k_b are positive (or) negative.

Hyperbolic point:

It a point at a Gaussian curvature is negative. (i.e) If $LN - M^2 < 0$ then the point ' P ' is called a Hyperbolic point.

$$\left[K < 0 \Leftrightarrow \frac{LN - M^2}{H^2} < 0 \Leftrightarrow LN - M^2 < 0, \text{ since } H^2 > 0\right]$$

$\therefore$ A point is a hyperbolic point $\Leftrightarrow$ the principal curvatures (κ_a & κ_B) at the points are of positive signs.

If point 'p ' is a jypthis hyperbolic $pt \Leftrightarrow K < 0$

$$\Leftrightarrow K_a \cdot K_b < 0$$
$\Leftrightarrow$ One of them is positive & the orther is negative

Parabolic point:

A point ' P ' the Gaussian curvature is zero (ie) If $LN - M^2 = 0$ then the point is called a parabolic point.

$$\left[K = 0, \Leftrightarrow \frac{LN - M^2}{H^2} = 0 \Leftrightarrow LN - M^2 = 0 \because H^2 > 0\right].$$

$\therefore$ A point is a parabolic point.

$\Leftrightarrow k_a \cdot k_b = 0$

$\Leftrightarrow$ atleast one of the principal curvature is zero.

Variation of normal curvatures with direction at the three types of points:

If κ_n is the normal curvatures at a point $P(u, v)$ in the direction (du, v) then,

$$\kappa_n = \frac{Ldu^2 + 2mdudv + vdv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Note that,

$$Edu^2 + 2Fdudv + Gdv^2 = \frac{1}{E}[E^2du^2 + 2EFdudv + EGdv^2].$$

$$= \frac{1}{E}[Edu + Fdv]^2 - F^2dv^2 + EGdv^2]$$

Since $E\ and\ EG - F^2 (= H^2)$ are assumed to be positive.

The denominator of the R.H.S of (1) is always positive.

$\therefore$ The sign of K_n depends upon the sign of the second fundamental form

$Ldu^2 + 2Mdudv + Ndv^2$

i) If ' P ' is an elliptic point then at ' P ' $LN - M^2 > 0 \therefore L \neq 0$

$$\therefore Ldu^2 + 2Mdudv + Ndv^2 = \frac{1}{L}[[Ldu + mdv]^2 + [LN - M^2]dv^2]$$

This shows that,

If $LN - M^2 > 0$ then the sign of $Ldu^2 + 2Mdudv + vdv^2 > 0$ (or) < 0 according as $L > 0$ (or) L<o.

Hence the sign of K_n maintains the same sign at an elliptic point for all directions

ii) If P is a parabolic point then at $P, LN - M^2 = 0 \therefore K_n$ maintains the Same sign for all direction that ' p ' excepts when $k_n = 0$.

iii) If p is a hyperbolic $'P$ 'then at ' P '.

$LN - m^2 < 0$

$\therefore k_n$ is positive for directions lying within a certain angle.

[The denominator of the R.H.S of (1) is always positive the second fundamental form $du^2 + 2Mdudv + Ndv^2$] negative for directions lying outside the angle & zero along the direction which form the angle.

Theorem:

Show that at an elliptic point the surface line entirely on outside of the tangent plane & at a hyperbolic point the surface cross over the tangent plane

Proof:

Let ' P ' be a point (u, v) on the given Surface & Let ' Q ' be a pt $(u + du, v + dv)$ in the neighborhood of P

If ' h ' is the length of the perpendicular from Q to the tangent plane to the surface at P then

$h = \frac{1}{2}[Ldu^2 + 2Mdudv + Ndv^2]$ ....(1)

If ' P ' is an elliptic point Then $Ldu^2 + 2Mdudv + vdv^2$ maintains the Same sign for all direction (du, dv) at ' P '.

Thus if ' P ' is an elliptic from (1), ' h ' has the same sign whether may be the position of Q.

Hence the entire surface lies on one side of the tangent plane at an elliptic point.

IF ' P ' is a hyperbolic point, then at ' P ', then $Ldu^2 + 2Mdudv + Ndv^2$ have ave as well as negative value.

$\therefore$ from (1),

If ' P ' is a hyperbolic point then ' h ' has positive as well as negative values.

$\therefore$ The surface near $'P$ ' lies on both sides of the tangent plane at P.

Thus the Surface Crosses over the tangent plane.

The Dupin Indicatrix:

If R_a, R_b to be the reciprocals of κ_a, κ_b then the curve of Section is the conic

$\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h,$

$z = 2b$. This conic is known as Dupin's Indicatrix.

It gives an immediate geometrical interpreter of the variation of normal curvature with direction.

Theorem:

If κ_a & κ_b are the principal curvature at ' O ' on the surface then the equation of the indicatrix is $\frac{x^2}{R_a}+\frac{y^2}{R_b}=2h\ldots\ldots\ldots.(1)z=h$, where $R_a=\frac{1}{k_a}$ & $R_b=\frac{1}{k_b}$

Proof:

Suppose, ' O ' is a given point on a given Surface.

Let Q be a point in the Dupin's indicatrix then Q is very near to ' O '.

[Then Q is a point on the curve of intersection of the surface & a plane which is parallel to the tangent plane at ' O ' & which interself The Surface in points very near to 'O'].

If ' h ' is the distance of D.T from the tangent plane at ' 0 '.

(i.e.) If ' h ' is the length of the perpendicular from a to the tangent plane at ' O '.

Then, $2h=Ldu^2+2Mdudv+Ndv^2$ ………….(2)

If we take the lines of curvature at ' O ' as parametric curves, then $F=0, M=0$.

∴ (2) ⇒

$2h=Ldu^2+Ndv^2$ ………… (3)

The Necessary condition. κ_n in the direction (du, dv) is given by,

$$\kappa_n=\frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$$

$$=\frac{Ldu^2+Ndv^2}{Edu^2+Gdv^2}\ \ldots\ldots\ldots(4)\ [\because F=m=0]$$

The direction ratios of the parametric curves $v=$ constant, $u=$ constant are $(1,0)\&(0,1)$

Since the line of curvature has been taken as the parametric curves from (4)

The principal curvature $k_a k_b$ are given by, $K_a=\frac{L(1)^2+N(0)^2}{E(1)^2+G(0)^2}=\frac{L}{E}\Rightarrow K_a=\frac{L}{E}\#(4)$

Similarly, $K_b=\frac{N}{G}$ ……….(5)

[For: -when $(du,dv)=(1{,}0)\Rightarrow k_n=k_a$

ill y when $(du,dv)=(0{,}1)\Rightarrow \kappa_n=\kappa_b$].

Using (5) in (3) we take, $2h=\kappa_a Edu^2+\kappa_b\mathrm{G}dv^2\ldots\ldots..(6)$

If ds_1 & ds_2 are alts. Of the arc length of the parametric curves $v=$ constant & $u=$ constant at 0

Then $ds_1=Edu^2$& $ds_2=Gdv^2$.

[$\because ds^2=Edu^2+2Fdudv+Gdv^2$ on $v=$ constant
$\Rightarrow dv=0$]

Therefore, (6) => $2h=\kappa_a ds_1^2+\kappa_b ds_2^2$

Take ' O ' as the 'origin

Let OX & OY of along the principal directions at 'O'& oz along the normal to the surface.

Let the co-ord of Q be (x, y, z) then $x = ds_1, y = ds_2 \& z = h$.

∴ from (7)

The equation of indicatrix is $z = h, 2h = \kappa_a x^2 + \kappa_b y^2$ … … … … (8) (or)

$$2h = \frac{x^2}{R_a} + \frac{y^2}{R_b}, z = h$$

where, $\underbrace{R_a = \frac{1}{K_a}}, R_b = \frac{1}{k_b}$

Radius of curvature at ' 0 '.

Note:

1) If K_a, K_b have the same sign the conic is an ellipse with semi axis of length $[2hR_a]^{1/2}$& $[2hR_b]^{1/2}$& is real (or) imaginary according to the sign of ' h '.

2.) If k_a & k_b have different signs of the conic is one of two conjugate hyperbolas.

In this case the directions of the asymptotes at ' O ' are called the asymptote directions at ' O '.

Example:

Prove that at any point ' P ' on a Surface there is a parabolic Such that the normal curvature of the Surface at ' P ' in any direction in the same as that of the paraboloid.

Proof:

We know that, the equation of the indicatrix is $\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h$ & $z = h$ ……….(1)

The equation to the surface for which (1) in the Coincide is obtained by eliminating ' R '.

$\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2z$ … … … … … (2)

(2) can be put in the parametric form as,

$$\bar{r} = (x, y, z) = \left(x, y, \frac{1}{2}\left[\frac{x^2}{R_a} + \frac{y^2}{R_b}\right]\right)$$

$$\therefore \overline{r_1} = \left(1,0, \frac{x}{R_a}\right) \& \bar{r}_2 = \left(0,1, \frac{y}{R_b}\right)$$

Here $\bar{r}_1 = \frac{\partial \bar{r}}{\partial x}$ & $\bar{r}_2 = \frac{\partial \bar{r}}{\partial y}$.

So, $E = 1 + \frac{x^2}{R_a^2}, F = \frac{2y}{R_a R_b} \& G = 1 + \frac{y^2}{R_b{}^2}$.

$$[\bar{r}_{11}, \bar{r}_1, \bar{r}_2] = HL = \frac{1}{R_a}$$

$$[\bar{r}_{12}, \overline{r_1}, \overline{r_2}] = HM = 0$$

$$\&[\bar{r}_{22}, \bar{r}_1, \bar{r}_2] = HN = \frac{1}{R_b}$$

Hence,

$$\kappa = \frac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}$$

$$= \frac{1}{H}\left\{\kappa_a \left[\frac{du}{ds}\right]^2 + \kappa_b \left[\frac{dv}{ds}\right]^2\right\}$$

Result:

The lines of curvature are in conjugate directions at every points.

Proof:

The directions of lines of curvature are given by,

$$[EM - FL]\frac{x^2}{m^2} + [EN - GL]\frac{x}{m} + [FN - GM] = 0$$

Thus $L\left[\frac{l_1 l_2}{m_1 m_2}\right] + M\left[\frac{l_1}{m_1} + \frac{l_2}{m_2}\right] + N$

$$= L\left\{\frac{FN - GM}{EM - LF}\right\} + M\left[\frac{GL - EN}{EM - FL}\right] + N$$

$$= \frac{LFN - LGM + MGL - MEN + NEM - NFL}{[EM - LF]}$$

$$= 0$$

(i.e.) $El_1 l_2 + F[l_1 m_2 + m_1 l_2] + Gm_1 m_2 = 0.$

∴ The lines of curvature are orthogonal.

Asymptotic Directions:

If Duplin indicatrix at ' 0 ' is a hyperbola, the direction of the asymptotic at ' 0 ' are called Asymptotic directions.

Asymptotic Lines:

An asymptotic line on a Surface is a curve whose direction at every pt on it is asymptotic equation of the asymptotic lines is,

Consider, $\frac{d\bar{r}}{ds}\frac{d\bar{N}}{ds} = \left[\bar{r}_1 \frac{du}{ds} + \bar{r}_2 \frac{dv}{ds}\right]\left[\bar{N}_1 \frac{du}{ds} + \bar{N}_2 \frac{dv}{ds}\right]$

$$= \bar{r}_1 \bar{N}_1 \left(\frac{du}{ds}\right)^2 + [\bar{r}_1 \bar{N}_2 + \bar{r}_2 \bar{N}_1]\frac{du}{ds}\cdot\frac{du}{ds} + \bar{r}_2 \bar{N}_2 \left(\frac{dv}{ds}\right)^2$$

$= \frac{Ldu^2 + 2Mdudv + N^2 dv^2}{ds^2}$ ……….(1)

But the direction at every point on the asymptotic lines is along the asymptotic direction the normal curvature at each pt. of the asymptotic lines is zero as

$Edu^2 + 2Fdudv + Gdv^2 \neq 0$ But $k_n = 0.$

$\therefore \Rightarrow Ldu^2 + 2Mdudv + N^2 dv^2 = 0$

$\therefore$ (1) $\Rightarrow \dfrac{d\bar{r}}{ds} \cdot \dfrac{d\bar{N}}{ds} = 0$

Asymptotic lines are self-conjugate Two directions (l_1, m_1) & (l_2, m_2) are conjugate if

$$Ll_1 l_2 + M[l_1 m_2 + m_1 l_2] + Gm_1 m_2 = a$$

But along the asymptotic lines the direction ratio are (du, dv).

$\therefore Ldu^2 + 2Mdudv + Ndv^2 = 0$. (by the above result).

Asymptotic lines are self-conjugate.

4.4. Developable:

A developable is a Surface enveloped by a one-parameter family of planes

A family (1-parameter family) is given by the equation $\bar{r} \cdot \bar{a} = p$

where $\bar{a}$ - represents the normal vector to the plane p - The length of the perpendicular from the origin ' O '.

Both $\bar{a}$ & p are functions of a real parameter ' u '.

Characteristic Line:

Let $\bar{r}.\bar{a} = p$ ……….(1)

be a family of planes & The planes $u, v (u < v)$ will intersect in a line provided they are not parallel.

Let $f(u) = \bar{r} \cdot \bar{a}(u) - p(u)$. then the eqn of the line of intersection is

$f(u) = 0$ &$f(v) = 0$.

$\therefore$ from Roller's Theorem, there exists a value $v_1{}^2$, such that $u < u_1 < v$ with $f(u_1) = 0$ as $v \to u, u_1 \to u$. and the equation of the limiting position of the character line in,

$\bar{r} - \bar{a} = p\ \bar{r} \cdot \dot{a} =$ p … … ….(2)

This line is also called the generators of the developable.

Characteristic points:

Consider the three planes u, v, w $(u < v < w)$ then there planes will generatly intersects with one pt & the limiting position of this pts as $v \to u$ and $u \to v$ independently is the characteristic points co responding to ' u '.

$\therefore$ By Rolle's the,

The equation determined is points are,

$$\left.\begin{aligned} \bar{r} \cdot \bar{a} &= p \\ \bar{r} \cdot \dot{\bar{a}} &= \dot{p} \\ \bar{r} \cdot \ddot{\bar{a}} &= \ddot{p} \end{aligned}\right\} \ldots\ldots\ldots(3)$$

If $\bar{a}, \dot{\bar{a}}, \ddot{\bar{a}}$ are $L \cdot D$, then these eqn's either have no solution or else the solution is indeterminate

Edge of Regression:

Consider the planes u, v, w be the chou. Pts corresponding to these three planes determine a curve on the developable this curve is called an edge of regression.

(i.e.) The char-pts corresponding to planes of the Family determine a curve on the developable called the edge of regression, with equation is given by,

$$\bar{r} \cdot \bar{a} = p$$
$$\bar{r} \cdot \dot{\bar{a}} = \dot{p}$$
$$\bar{r} \cdot \dot{a} = \dot{p}$$

$\bar{r} \cdot \dot{a} = \dot{P}$, where $\bar{r}$ is regarded as a function of u.

Result:

The tangent line to the edge of representation is parallel to the characteristic line

Proof:

The equation of the edge of regression is, $\bar{r} \cdot \bar{a} = p \; \dots \dots \dots . (1)$

$\bar{r} \cdot \dot{a} = \dot{p} \dots \dots \dots (2)$

$\bar{r} \cdot \ddot{a} = \ddot{p} \dots \dots \dots . (3)$

Let$t$ be the tangent vector to the edge of regression

Diff (1) with respect to ' u '

$$\left[\frac{d}{ds}(\bar{r}) \cdot \frac{ds}{du}\right] \cdot \bar{a} + \bar{r} \cdot \dot{a} = \dot{p}$$
$$[\dot{S}\mathrm{t}] \cdot \bar{a} + \bar{T} \cdot \dot{a} = \dot{p}$$
$$\dot{S}t \cdot \bar{a} + \dot{p} = \bar{p}[\mathrm{by}(2)]$$
$$\Rightarrow (\dot{S}\bar{t}) \cdot \bar{a} = 0$$
$$\Rightarrow \; \bar{t}. \bar{a} = 0 \dots \dots \dots \dots . (4)$$

Diff (2) w.r. to ' u '

$$\left[\frac{d}{ds}(\bar{r})\frac{ds}{du}\right]\dot{a} + \bar{r} \cdot \ddot{a} = \ddot{p}$$
$$\dot{S}\bar{t} \cdot \dot{\bar{a}} + \bar{r} \cdot \dot{\bar{a}} = \ddot{p}$$
$$\Rightarrow \dot{S}\bar{t} \cdot \dot{\bar{a}} + p = \ddot{p}[by(3)]$$
$$\Rightarrow \dot{s}\bar{t} \cdot \dot{\bar{a}} = 0$$
$$\Rightarrow \bar{t} \cdot \dot{\bar{a}} = 0 \; \dots \dots \dots (5)$$

From (4) & (5) we get

$\bar{t}$ is perpendicular to $\bar{a}$ & $\dot{a}$

$\bar{t}$ is parallel to $\bar{a} \times \dot{\bar{a}}$ …………(6)

We know that, the line intersection of (1) & (2) is the characteristic line.

$\therefore$ It is parallel to $\bar{a} \times \dot{\bar{a}}$ ....(7)

From (5) & (7) we get

$\bar{t}$ is parallel to the characteristic line

Result :

A developable consists of two sheets which are tangent to the edge of the regression along a sharp edge.

Proof :

Let ' c ' be the edge of regression

& Let ' O ' be a pt on the edge of regression at which $s = 0$.

Let ox, oy & oz be a set of orthogonal traid [rectangular Cartesian axis] chosen respectively along $\bar{t}, \bar{n}, \& \bar{b}$ at ' 0 '. then,

Any point on the develpatle has position vector given by $\bar{R} = \bar{r} + v\bar{t}$

$\bar{R}$ also can be written as,

$\bar{R} = \overline{O'P} = \overline{O'Q} + \overline{QP} = \bar{\gamma}(s) + v\bar{t}(s)$

(on extending $\bar{R}$ in powers of ' s ')

$$\bar{R} = \bar{r}(s) + v\bar{t}(s)$$

$$= \bar{r}(0) + s\bar{r}'(0) + \frac{s^2}{2!}\bar{r}''(0) + \frac{s^3}{3!}\bar{r}'''(0) + o(s^4)$$

$$+ v\left[t(0) + s\bar{t}'(0) + \frac{s^2}{2!}\bar{t}''(0) + \frac{s^3}{3!}t'''(0) + o(s^4)\right]$$

.... (2)

$$\therefore \bar{R} = \left[0 + s\bar{t} + \frac{s^2}{2}[k\bar{n}] + \frac{s^3}{6}\left[k'\bar{n} + k(\tau\bar{b} - k\bar{t})\right] + o(s^4)\right.$$

$$+v\left[\bar{t} + sk\bar{n} + \frac{s^2}{2}\left[k'\bar{n} + k(\tau\bar{b} - k\bar{t}) + o(s^3)\right]\right]$$

$$= \left[s - \frac{s^3}{6}k^2 + v - \frac{vs^2k^2}{2}\right]\bar{t} + \left[\frac{s^2k}{2} + \frac{s^3}{6}k' + vsk + \frac{s^2}{2}k'\right]n + \left[\frac{s^3k\bar{\tau}}{6} + \frac{s^2}{2}k\tau\right]\bar{b}$$

The normal plane $x = 0$, meets the developable surface where,

$$s-\frac{s^3}{6}k^2+v-\frac{vs^2k^2}{2}=0$$

$$\Rightarrow v\left[1-\frac{s^2k^2}{2}\right]=\frac{s^3k^2}{6}-s$$

$$\Rightarrow v=\left[\frac{s^2k^2}{6}-s\right]\left[1-\frac{s^2k^2}{2}\right]-1$$

$$=\left[\frac{s^3k^2}{6}-s\right]\left[1+\frac{s^2k^2}{2}\right][\because (1-x)^{-1}=1+x+x^2+\cdots]$$

$$=-s-\frac{s^3k^2}{2}+\frac{s^3k^2}{6}+\frac{s^5k^4}{12}$$

$$=-s-\frac{2s^3k^2}{6}+o(s^4)$$

$V=-S-\frac{s^3k^2}{6}$ Sub in (2) & then comparing with (1)

$$y=\frac{-1}{2}ks^2+o(s^2)$$

$$z=-\frac{1}{3}k\tau s^3+o(s^4)$$

eliminating ' s ' between these eqns

$$z^2=\frac{-8}{9}\frac{\tau^2}{k}y^3$$

From this equation it follows that, the developable cuts the normal plane to the edge of regression in a cusp whose tangent is along the principal normal.

The two sheets of the developable are thus tangent to the edge of regression along a Sharp edge.

Result:

The tangent plane is same all the pts of the generators of a developable surface

Proof:

Let $\bar{r}=\bar{r}(s)$ be the eq. of the edge of regression & $\bar{t}$ be the unit tangent vector at any pt of the edge of regression .

We know that, the equation, of the developable is,

$$\overline{OQ}=\overline{\delta P}+\overline{PQ}$$

$\bar{R}=\bar{r}\,(s)+v\,\bar{t}(s)$ ....(1)

Diff (1) w.r. to V&S (partial)

$$\frac{\partial R}{\partial v}=\overline{R_1}=\bar{\mathrm{t}}$$

$\& \frac{\partial \bar{R}}{\partial S} = \bar{R}_2 = \bar{t} + vk\bar{n}$

$\therefore R_1 \times R_2 = \bar{t} \times [\bar{t} + v\kappa\bar{n}]$

$= 0 + Vk(\bar{t} \times \bar{n})$

(i.e.) $H\bar{N} = V\kappa\bar{b}$

$\Rightarrow \bar{N} = \bar{b}$

$\Rightarrow \bar{N}$ is a function of 's' only '

$\Rightarrow \bar{N}$ is independent of v.

$\Rightarrow \bar{N}$ is Some at every point along a curve where ' s ' is fixed.

$\Rightarrow \bar{N}$ is Some at every pt of char line

$\Rightarrow \bar{N}$ is Some at every pt generator of the developable.

$\Rightarrow$ Tangent plane at every pt generator of the developable.

Result:

The osculating plane of the edge of regression at any point ' p ' in the tangent plane to the developable Surface at P.

Proof:

The equation of the edge of regression is

$\bar{r} - \bar{a} = p$ ……….. (1)

$\bar{r} \cdot \dot{\bar{a}} = \dot{p}$ ……….. (2)

$\bar{r} \cdot \ddot{a} = \ddot{p}$ ……….. (3)

Diff (1) w. r. to ' u '

$\dot{r} \cdot \bar{a} + \bar{r} \cdot \dot{\bar{a}} = \dot{p}$

(i.e.) $\dot{r}, \bar{a} = 0$ ……….(4)

Diff (2) w.r.to ' u'.

$\dot{r}\dot{\bar{a}} + \dot{r} \cdot \ddot{a} = \ddot{p}$

$\dot{r} + \dot{a} + \ddot{p} = \ddot{p}$(by (3))

Diff (4) w.r.to 'u'

$\ddot{r} \cdot \bar{a} + \dot{r} - \dot{a} = 0$

(ie) $\ddot{r} \cdot \bar{a} + 0 = 0$ by (5)

$\ddot{r} \cdot \bar{a} = 0$ …………. (6)

From (4) & (6) we get,

$\left.\begin{matrix} \dot{r}\bar{a} = 0 \\ \ddot{r}\bar{a} = 0 \end{matrix}\right\} \Rightarrow \bar{a}$ is parallel to $\bar{r} \times \ddot{\bar{r}}$

But $\bar{r} \times \ddot{\bar{r}}$ parallel to $\bar{t} \times \bar{n} = \bar{b}$

So $\bar{a}$ is parallel to $\bar{b}$

$\Rightarrow$ $\bar{a}$ is perpendicular to the plane ....(7)

But $\bar{a}$ is normal to the corresponding plane $\bar{t}$ in the Family.

Thus the tangent to the developable surface is Same as the plane of edge of regression.

4.5.Developables associated with space curves:

Osculating Developable:

The one-parameter family of osculating planes at pts on a skew curve form the "osculating developable" of the curve.

Result:

The edge of regression in the curve itself of the osculating developable. (or)

The generators of osculating developable of a space curve are the tangent to the curve & the edge of regression of the osculating developable of a space curve in the curve itself.

Proof:

Let $\bar{r} = \bar{r}(s)$ be the given curve.

The family of osculating planes has equation $[\bar{R} - \bar{r}(s)] \cdot \bar{b} = \bar{0}$ ....(1)

where, $\bar{R}$ is the position vector of an arbitrary point on the plane.

& $\bar{r}$ in the point on the curve

Differentiate w.r.to 'S'.

$(\bar{R} - \bar{r})' \cdot \bar{b}' + (-\bar{r}') \cdot \bar{b} = 0$

(i.e.) $(\bar{R} - \bar{r}) \cdot (-\bar{\tau}) + (-\bar{t}) \cdot \bar{b} = 0. [\because \tau \neq 0]$

(ie)$(\bar{R} - \bar{r}) \cdot \bar{n} = 0$ ....(2)

Diff (2) w.r.to ' s '

$$(-\bar{r}')\bar{n} + (\bar{R} - \bar{r}) \cdot \bar{n}' = 0.$$
$$-\bar{t} - \bar{n} + (\bar{R} - \bar{r})(\tau\bar{b} - k\bar{t}) = 0$$
$$(\bar{R} - \bar{r}) \cdot \tau\bar{b} - (\bar{R} - \bar{r})k\bar{t} = 0$$
$$0 - (\bar{R} - \bar{r})k\bar{t} = 0 \quad [\because \text{ by } (1)\ \ \tau \neq 0 \& k \neq 0]$$

$(\bar{R} - \bar{r})k\bar{t} = 0$ ....(3)

From (1), (2), (3) we see that, $(\bar{R} - \bar{r})$ is $\perp^r$ to $\bar{t}, \bar{n}, \bar{b}$

$\therefore \bar{R} - \bar{r} = 0$

$\bar{R} = \bar{r}$

$\Rightarrow$ The edge of regression is the curve itself.

Polar Developable:

The family of normal planes to a skew curve form the "polar Developable" of the given curve

Result :

The edge of regression of the polar developable in the locus of centers Spherical curvature of the given curve.

Proof :

Let the equation of the curve to $\bar{r} = \bar{r}(s)$

The equation of the family of normal planes, $(\bar{R} - \bar{\gamma}) \cdot \bar{t} = \bar{0}$ ....(1)

Differentiate w.r.to ' s '

$(\bar{R} - \bar{r})\bar{t}' + (-\bar{r}')\bar{t} = 0.$

$(\bar{R} - \bar{r})(k\bar{n}) - \bar{t} \cdot \bar{t} = 0$

$(\bar{R} - \bar{r})k\bar{n} - 1 = 0$

(i.e.) $(\bar{R} - \bar{r})\kappa\bar{n} = 1$

(i.e.) $(\bar{R} - \bar{r})\bar{n} = \frac{1}{\kappa} = \rho$ ....(2)

Diff (2) w.r. to ' s '

$(\bar{R} - \bar{r})\bar{n}' + (-\bar{r}')\bar{n} = \rho'$

$(\bar{R} - \bar{r})(\tau\bar{b} - k\bar{t}) - \bar{t} \cdot \bar{n} = \rho'$

$\tau(\bar{R} - \bar{r})\bar{b} - k(\bar{R} - \bar{r})\bar{t} - 0 = \rho'$

$\tau(\bar{R} - \bar{r})\bar{b} - 0 - 0 = \rho'$ [by (1)].

(ie) $\tau(\bar{R} - \bar{r})\bar{b} = \rho'$

(ie) $(\bar{R} - \bar{r})\bar{b} = \frac{\rho'}{\tau} = \sigma\rho'$ ....(3)

$\therefore$ from eqn (1) we see that, $(\bar{R} - \bar{r})$ is perpendicular to ' $\bar{t}$ '.

$\therefore$ $(\bar{R} - \bar{r})$ lies on the normal plane $\therefore$ $\bar{R} - \bar{r} = \lambda\bar{b} + \mu\bar{n}$ ....(4)

Taking (.) product to (4) with $\bar{b}$,

$(4) \Rightarrow (\bar{R} - \bar{r}) \cdot \bar{b} = \lambda\bar{b} \cdot \bar{b} + \mu\bar{n} \cdot \bar{b}$

$\Rightarrow (\bar{R} - \bar{r}) \cdot \bar{b} = \lambda + 0$

$=> \sigma\rho' = \lambda$ ... ... $(*)[by 3]$

Taking (.) product to (4) with $\bar{n}$, (4)

$\Rightarrow (\bar{R} - \bar{r})\bar{n} = \lambda\bar{b} \cdot \bar{n} + \mu\bar{n} \cdot \bar{n}$

$(\bar{R} - \bar{r}) \cdot \bar{n} = 0 + \mu$

(i.e.) $\rho = \mu$ ... . $(**)$(by (2))

Sub (*) & (**) in (4)

$\therefore$ $(4) \Rightarrow \bar{R} - \bar{r} = \sigma\rho'\bar{b} + \rho\bar{n}$

(or) $\bar{R} = \bar{r} + \sigma\rho'\bar{b} + \rho\bar{n}$

$\therefore$ $\bar{R}$ in the position vector of the centre of Spherical curvature.

(ie) which is the locus of cube of Spherical curvature.

Rectifying Developable:

The rectifying planes to a Skew curve determine the "Rectifying developable" of the given curve.

Result:

The curve is a geodesic in the rectifying developable.

Proof:

Let $\bar{r} = \bar{r}(s)$ be the given curve

then the char line of the rectifying developable is in by $(\bar{R} - \bar{r}) \cdot \bar{n} = 0$ ....(1)

Diff w. to ' s '.

$$(\bar{R} - \bar{r})\bar{n}' + (-\bar{r}) \cdot \bar{n} = 0.$$
$$(\bar{R} - \bar{r})(\tau\bar{b} - k\bar{t}) - \bar{t} - \bar{n} = 0$$

$(\bar{R} - \bar{r})\left(\tau\bar{b} - k\bar{t}\right) = 0 \ldots \ldots \ldots \ldots (2)$

Let $\overline{R_1}$ denote the diff of $\bar{R}$ $w.r.$ to μ.

& $\overline{R_2}$ denote the diff of $\bar{R}$ $w.r.$ to 's'.

$$\therefore \bar{R}_1 = \tau\bar{t} + k\bar{b}$$
$$\&\bar{R}_2 = \bar{t}(1 + \tau'\mu) + \mu k'\bar{n}$$

$$\therefore \overline{R_1} \times \overline{R_2} = \left(\tau\bar{t} + k^{\bar{b}}\right) + x(1 + \mu\tau')\bar{t} + \mu k'\bar{n}\Big)$$
$$= 0 + \mu\tau k'(\bar{t} \times \bar{n}) + k(1 + \mu\tau')(\bar{b} \times \bar{t}) + \kappa\,\mu k'(\bar{b} \times '$$
$$= \mu\tau\kappa'\bar{b} - k(1 + \mu\tau')\bar{n} - k'\mu k'\bar{t}$$
$$= 1$$
$$= \kappa(1 + \tau'\mu)\bar{n} - \mu\tau k'\bar{n}$$

$\overline{R_1} \times \overline{R_2} = H\bar{N}$ is parallel to $\bar{n}$

⇒ The curve is a geodesic on rectifying developable.

Note:

1. Any developable, which is not cylinder (or) a cone, may be regarded as the osculating developable of its edge of regression.
2. The equation of principal planes namely osculating plane, normal plane & rectifying plane of a space curve at a point P contains only a single parameter, which is usually taken as arc length.

   Therefore, Their envelopes are developable & They are Osculating developable, polar developable & rectifying developable.

   The generators of polar & rectifying developable are called polar and rectifying lines respectively.

Example 1:

Prove that the edge of regression of the rectifying developable has eqn.

$\bar{R} = \bar{r} + k\frac{[\tau\bar{t}+k\bar{b}]}{[k'\tau-k\tau]}$ (or)

The Rectifying plane to a skew curve determine the rectifying developable of the given curve. Prove that the edge of regression of the rectifying developable has equation,

$$\bar{k} = \bar{r} + \frac{k[\tau\bar{t}+k\bar{b}]}{[k'\tau-k']}$$

Proof:

Let the equation of the curve be $\bar{r} = \bar{r}(s)$ equation of the rectifying plane is,

$(\bar{R}-r)\cdot\bar{n} = 0$

Diff (1) w.r.to ' s '.

$$(\bar{R}-\bar{r})\cdot\bar{n}' + (-\bar{r}')\bar{n} = 0$$

Diff (2) w.r.to 's'.

$$(-\bar{r}')(\tau\bar{b}-k\bar{t}) + (\bar{R}-\bar{r})\left[\tau\bar{b}' + \tau'\bar{b} - k'\bar{t}' - k'\bar{t}\right]$$
$$-\bar{t}'\bar{n} - \bar{n}'\bar{t} = 0$$

(i.e.) $(-\bar{t})(\tau\bar{b}-k\bar{t}) + (\bar{R}-\bar{r})\left[\tau(-\tau\bar{n}) + \tau'(\bar{b}) - k(k\bar{n}) - k'\bar{t}\right]$

$$-(k\bar{n})\cdot\bar{n} - (\tau\bar{b}-k\bar{t})\cdot\bar{t} = 0$$

(i.e.) $0 + k + (\bar{R}-\bar{r})\left[-\tau^2\bar{n} + \tau'\bar{b} - k^2\bar{n} - k'\bar{t}\right] - \kappa - 0 + k = 0$

(i.e.) $k + (\bar{R}-\bar{r})\left[-(\tau^2+k^2)\bar{n} + \tau'\bar{b} - k'\bar{t}\right] = 0$

$k - (\bar{R}-\bar{r})(\tau^2+k^2)\bar{n} + (\bar{k}-\bar{r})(\tau'\bar{b}-k'\bar{t}) = 0$

$k + (\bar{R}-\bar{r})(\tau'\bar{b}-k'\bar{t}) = 0$ - (4) $[\because (\bar{R}-\bar{r})\cdot\bar{n} = 0]$

$\therefore$ eqn (1) & (2) Shows that $\bar{R}-\bar{r}$ is parallel to both $\bar{n}$ & $(\tau\bar{b}-k\bar{t})$

$\bar{R}-\bar{r}$ is parallel to $\bar{n}\times(\tau\bar{b}-k\bar{t})$

(i.e.) It is parallel to $\tau\bar{t}+kb$

$\Rightarrow (\bar{R}-\bar{r}) = \mu(\tau\bar{t}+k\bar{b})$ (5), μ is a scalar.

Sub the values of $\bar{R}-\bar{r}$ from (5) in (4).

$\therefore$ (4) $\Rightarrow k + \mu(\tau\bar{t}+\kappa\bar{b})\cdot(\tau'\bar{b}-k'\bar{t}) = 0$

$$k + \mu(0 - k'\tau + k\tau' - 0) = 0$$
$$k + \mu(k\tau' - k'\tau) = 0$$

(i.e.) $\mu = \frac{-k}{(k\tau'-k'\tau)}$

(ie) $\mu = \frac{k}{(\kappa'\tau-\kappa\tau')}$ sub in (5)

$\therefore$ (S) $\Rightarrow \bar{R} - \bar{r} = \frac{k}{(k'\tau - k\tau')}(\tau\bar{t} + k\bar{b})$

(ie) $\bar{R} = \bar{r} + \frac{k[\tau\bar{t}+k\bar{b}]}{[k^1\tau - k\tau']}$

Theorem 1:

A necessary and sufficient condition for a surface to be a developable is that its Gaussian curvature be zero.

Proof :

Necessary part:

Let the surface be a developable

To prove that the Gaussian curvature $k = 0$

If the developable surface is a cylinder (or) a Cone, then the Gaussian curvature is evidently zero. If these cases are excluded, then the developable may be regarded as the osculating developable of its edge of regression & its equation may be written as,

$\bar{R} = \bar{r}(s) + v\bar{t}(s)$

(i.e.) $\bar{R} = \bar{r} + V(\bar{t})$ ....(1)

Let the Suffices 1 & 2 denote the diff w.r. to 'S' & 'V' respectively.

we know that

Gaussian curvature $k = \frac{LN - M^2}{EG - F^2}$ ....(A)

where $\left.\begin{aligned} L &= \bar{N}\cdot\bar{R}_{11} \\ \text{where } \bar{N} &= \frac{\bar{R}_1 \times \bar{R}_2}{H} \\ M &= \bar{N}\cdot\bar{R}_{12} \\ N &= \bar{N}\cdot\bar{R}_{22} \end{aligned}\right\}$ ....(B)

$[\because R = r]$

Diff (1) w.r.to 's'.

$\bar{R}_1 = \bar{r}' + v\bar{t}'$

$R_1 = \bar{t} + v'k'\bar{n} \Rightarrow R_{11} = \bar{t}' + vk'\bar{n} + vk'\bar{n}'$

(ie) $R_{11} = k\bar{n} + k'\bar{n} + vk(\tau\bar{b} - k\bar{t})$

Dift (1) w.r. to 'v ' & $R_{12} = 0 + k\bar{n} \Rightarrow R_{12} = k\bar{n}$

$\bar{R}_2 = 0 + \bar{t}$

$\Rightarrow \bar{R}_2 = \bar{t} \Rightarrow \bar{R}_{22} = 0$

& $\bar{R}_{21} = \bar{t}' = k\bar{n} \Rightarrow \bar{R}_{21} = k\bar{n}$

(B) $\Rightarrow \therefore L = \bar{N}\cdot\bar{R}_H = -\frac{Vk\bar{b}}{H}[k\bar{n} + Vk\cdot\bar{n} + Vk(\tau\bar{b} - k\bar{t})]$

$L = \frac{-v^2 k^2 \tau}{H}.$

$\therefore M = \bar{N} \cdot \bar{R}_{12} = \frac{-vk\bar{b}}{H}[k\bar{n}] = 0$

$M = 0$

$\therefore N = \bar{N} \cdot \bar{R}_{22} = \frac{-Vk\bar{b}}{H}(0) = 0$

$\Rightarrow N = 0$

$(A) \Rightarrow K = \frac{LN - M^2}{EG - F^2}$

$= \frac{\left(\frac{-V^2K^2\tau}{H}\right)(0) - 0^2}{EG - F^2}$

Gaussian curvature $\Rightarrow k = 0$

Sufficient part:

Assume that $k = 0$

To prove that: The surface is developable.

given, $k = 0 \Rightarrow \frac{LN-M^2}{EG-F^2} = 0.$

$\Rightarrow LN - M^2 = 0$
$\Rightarrow (\bar{r}_1 \cdot \bar{N}_1)(\bar{r}_2 \cdot \bar{N}_2) - (r_1 N_2)(\bar{r}_2 \cdot \bar{N}_1) = 0$
$\Rightarrow (\bar{r}_1 \times \bar{r}_2)(\bar{N}_1 \times \bar{N}_2) = 0$

$H\bar{N}\left(\overline{N_1} \times \bar{N}_2\right) = 0$

$\bar{N}(\bar{N}_1 \times \bar{N}_2) = 0 \; [\because H \neq 0]$

(ie) $[\bar{N}, \bar{N}_1, \bar{N}_2] = 0$ ……….(1)

from (1), we have any one of the following possibility,

i) $\bar{N}, \bar{N}_1, \bar{N}_2$ are coplanar.

ii) $\bar{N}_1 = 0$

iii) $\bar{N}_2 = 0$

iv) $\bar{N}_1 = \mu\bar{N}_2$

Case (i):

Since $\bar{N}$ is a vector of unit length of $\bar{N} \cdot \bar{N} = 1$ we have $2\bar{N}_1 \cdot \bar{N}_2$ (or) $\bar{N}_1 \cdot \bar{N} = 0$

Similarly, $\bar{N}_2 \cdot \bar{N} = 0$

(ie) $\bar{N}$ is perpendicular to both $\bar{N}_1 \& \bar{N}_2$

(ie) $\bar{N}, \bar{N}_1, \bar{N}_2$ cant be coplanar.

Case (ii): $\bar{N}_1 = 0.$

The eqn. of the tangent plane at any point of

$\bar{r}(u,v)$ is given by $(\bar{R}-\bar{r})\bar{N}_1 = 0$. isithue

$\therefore$ we have,

$$\frac{\partial}{\partial u}\{(\bar{R}-\bar{r})\bar{N}\} = -\bar{r}_1\bar{N} + (\bar{R}-\bar{r})\bar{N}_1$$

$$= 0 + 0 = 0$$

$\Rightarrow (\bar{R}-\bar{r})\bar{N}$ depends only on V the surface is the envelope of one parameters family of planes.

Hence it is developable.

Case (iii):

$\bar{N}_2 = 0$

proceeding similarly as in (ii) we see that the eqn of the tangent plane contains only one parameter ' u '.

Hence in this case also the surface is developable

Case (iv):

$\bar{N}_1 = \mu\bar{N}_2$

Let us change the parameter (u,v) to (u',v') by the transformation

$u = u' + v' \,\&\, v = u' - \mu v'$

then we obtain,

$$\bar{N}_1' = \frac{\partial \bar{N}}{\partial u}\cdot\frac{\partial u}{\partial u'} + \frac{\partial \bar{N}}{\partial v}\cdot\frac{\partial v}{\partial u'}$$

$$= \bar{N}_1\cdot 1 + \bar{N}_2\cdot 1$$

$$\bar{N}_1' = \bar{N}_1 + \bar{N}_2$$

$$\&\bar{N}_2' = \frac{\partial \bar{N}}{\partial v'} = \frac{\partial \bar{N}}{\partial u}\cdot\frac{\partial u}{\partial v'} + \frac{\partial \bar{N}}{\partial v}\cdot\frac{\partial v}{\partial v'}$$

$$= \bar{N}_1\cdot 1 + \bar{N}_2(-\mu)$$

$$= \bar{N}_1 - \mu\bar{N}_2$$

$$\bar{N}_2' = 0$$

These relation shows that N, the surface normal depends on only one parameter

Hence by case (iii)

The surface is developable.

4.6.Developable Associated with curves on Surfaces:

Theorem 1:Monge's Theorem

A necessary and sufficient condition that a curve on a surface roving be a line of curvature is that the surface normal along the curve form a developable.

Proof:

We consider the surface formed by the normal along the curve $\bar{r} = \bar{r}(s)$

Any point on this surface will have the position Vector, $\bar{R} = \bar{r}(s) + v\bar{N}(s)$ ………..(1)

where ' S ' and ' v ' are the parameters

Let the suffixes 1 & 2 denote diff w.r. to 's' & 'v' respectively.

From (1), we have,

$\bar{R}_1 = \bar{r}_1' + v\bar{N}'$ (Diff w.r. to 's')

(i.e.) $\bar{R}_1 = \bar{t} + v\bar{N}'$

& $\bar{R}_{12} = \bar{N}' = R_{21}$ (Diff w.r. to ' v')

$\overline{R_2} = \bar{N}$ (Diff w.r. to 'v')

&$\bar{R}_{22} = 0$ (w.r. to ' v')

$\therefore \bar{N} = \dfrac{\bar{R}_1 \times \bar{R}_2}{H}$

$\therefore M = \bar{R}_{12} \cdot \bar{N} = \bar{R}_{12} \cdot \dfrac{[\overline{R_1} \times \bar{R}_2]}{H}$

(i.e.) $HM = \overline{R_{12}}[\overline{R_1} \times \overline{R_2}]$

(ie) $HM = [\overline{R_{12}}, \overline{R_1}, \bar{R}_2]$

Similarly, $HN = [\overline{R_{22}}, \overline{R_1}, \overline{R_2}]$

$= [\bar{t} + v\bar{N}', \bar{N}', \bar{N}] = 0$
$= [\bar{t}, \bar{N}', \bar{N}] + [v\bar{N}', \bar{N}', \bar{N}]$

$HM = [\bar{t}, \bar{N}, \bar{N}]$ ………..(3)

Hence the Gaussian curvature ' k ' is given by, $K = \frac{LN-M^2}{H^2} = \frac{-M^2}{H^2}$ ………..(4)

We know that, the surface is developable,

$\Leftrightarrow K = 0$

$\Leftrightarrow \dfrac{-M^2}{H^2} = 0$

$\Leftrightarrow M = 0$

$\Leftrightarrow [\bar{t}, \bar{N}, \bar{N}'] = 0.$

Hence it is enough if we prove The curve $\bar{r} = \bar{r}(s)$ is a line of curvature (or) the surface $\bar{r} = \bar{r}(u, v) \Leftrightarrow [\bar{t}, \bar{N}, \bar{N}^l] = 0.$

Necessary part :

Let$[\bar{t}, \bar{N}, \bar{N}'] = 0$ ....(5)

T.P.T: The curve $\bar{r} = \bar{r}(s)$ a line of curvature on the surface $\bar{r} = \bar{r}(u, v)$

$\therefore$ (5) $\Rightarrow (\bar{t} \times \bar{N}') \cdot \bar{N} = 0$

Here $\bar{N} \neq 0$.

&$\bar{t} \times \bar{N}'$ is not perpendicular to $\bar{N}$

$\therefore \bar{t} \times \bar{N}' = 0$

(i.e.) $\bar{t}$ is $||^{el}$ to $\bar{N}'$

$\Rightarrow \bar{N}' = -k\bar{t}$ for some function k

$$\Rightarrow \frac{d\bar{N}}{ds} = -K\frac{d\bar{r}}{ds}$$

(i.e.) $d\bar{N} + kd\bar{r} = 0$

$\Rightarrow$ The given curve is a line of curvature

(by Rodrisue's formula)

Sufficient part:

Let the curve $\bar{r} = \bar{r}(s)$ be a line of curvature

Then $d\bar{N} + kd\bar{r} = 0$

(ie) $\frac{d\bar{N}}{ds} = -k\frac{d\bar{r}}{ds}$

(ie) $\bar{N}' = -k\bar{t}$ ....(6)

$\therefore [\bar{t}, \tilde{N}, N'] = [\bar{E}, \bar{N}, -k\bar{t}]$

$= 0$

Theorem 2:

Let $\bar{r} = \bar{r}(u, v)$ be a Surface &$\bar{r} = \bar{r}(s)$ be a curve ' c ' on it. The tangent planes at point on c lying on a Surface form a developable. Then the char. line of the developable at any point ' p ' on ' c ' is in a direction conjugate to that of the tangent to ' c ' at ' p '.

Proof :

We know that, the equation of the families of tangent plane is $(\bar{R} - \bar{r})\bar{N} = 0$ ....(1)

Diff (1) w.r.to ' s ' we get

$(\bar{R} - \bar{r})\frac{d\bar{N}}{ds} = 0$ .... (2)

Let (l, m) be the direction co-eff of the char line at P then

$(\bar{R} - \bar{r}) = lr_1 + mr_2$ ... ... (3)

$\therefore$(2) $\Rightarrow (lr_1 + mr_2)(N_1u' + r_2v') = 0$

$\Rightarrow Llu' + m(lv' + mu') + Nmv' = 0$ is the direction (l, m) is conjugate to the direction(u', v') of the tangent to ' c ' at ' p '.

4.7.Minimal Surface:

Surfaces whose mean curvatures is zero at all the points are called the minimal surfaces.

Theorem 1:

If there is a surface of minimum area com passing through a closed curve it is necessary a minimal surface that is a surface of zero mean curvature.

Proof:

Let ε be a surface bounded by a closed curve ' c ' and let ε ' be another surface normal let ε_1 and ε_2 the both small.

(ie) $\varepsilon_1 = O(\varepsilon)$ $\epsilon_2 = \alpha(\epsilon)$ as $\epsilon \longrightarrow 0$.

Then if $\bar{R}$ denotes the position vector of the displaced surface we have

$$\bar{R} = \bar{r} + \epsilon\hat{N}$$
$$\frac{\partial \bar{R}}{\partial u} = \bar{R}_1 = \bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1$$
$$\frac{\partial \bar{R}}{\partial v} = \bar{R}_2 = \bar{r}_2 + \epsilon_2\hat{N} + \epsilon\bar{N}_2$$

Lot E^*, F^*, G^* denote the first fundamental Coefficients of E'

Then $E^* = \overline{R_1} \cdot \bar{R}_1$

$$= \bar{R}_1 \cdot \bar{R}_1$$
$$= (\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1) \cdot (\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1)$$
$$= \bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1) \cdot (\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1)$$
$$= (\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1)(\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1)$$
$$= E - 2\epsilon L 0(\epsilon^2)$$

$$F^* = \bar{R}_1 \cdot \bar{R}_2$$
$$= (\bar{r}_1 + \epsilon_1\hat{N} + \epsilon\bar{N}_1)(\bar{r}_2 + \epsilon_2\hat{N} + \epsilon\bar{N}_2)$$
$$= \bar{r}_1 \cdot \bar{r}_2 + (\bar{r}_1 + \bar{N}_2 + \bar{r}_2\bar{N}_1)\epsilon + o(\epsilon^2)$$
$$= \bar{r}_1 \cdot r_2 + (\bar{r}_1 \cdot \bar{N}_2 + \bar{r}_2 \cdot \bar{N}_1)\epsilon + o(\epsilon^2)$$
$$= F + (-M, -M)\epsilon + o(\epsilon^2)$$
$$= F - 2M\epsilon + 0(\epsilon^2)$$

$$G^* = \bar{R}_2 \cdot \bar{R}_2$$
$$= (\bar{r}_2 + \epsilon_2\hat{N} + \epsilon\bar{N}_2)(\bar{r}_2 + \epsilon_2\hat{N} + \epsilon\bar{N}_2)$$
$$= \bar{r}_2 - \bar{r}_2 + (\bar{r}_2 \cdot \bar{N}_2 + N_2 \cdot \bar{r}_2)\epsilon + O(\epsilon^2)$$
$$= G + (-N \cdot -N)\epsilon + o(\epsilon^2) \text{ as } \epsilon \to 0$$
$$= G - 2N\epsilon + o(\epsilon^2) \text{ as } \epsilon \to 0.$$

Then $H^{*2} = E^*G^* - F^{*2}$.

$=[E-2\epsilon L+o(\epsilon_2^2)][G-2\epsilon N+o(\epsilon^2)]$

$[F-2M\epsilon+o(\epsilon^2)]^2$

$=EG-F^2-2\epsilon[EN+GL-2FM]+o(\epsilon^2)$

$=(EG-F^2)[1-2\epsilon\cdot 2\mu]+o(\epsilon^2)$

where $\mu=$ mean curvature.

$$=\frac{EN+GL-2FM}{2(EG-F^2)}$$

(i.e.) $H^{*2}=H^2(1-4\in M)+O(\epsilon^2)$ as $\epsilon\to 0$

$H^{*}=H(1-4\epsilon\mu)^{1/2}+O(\epsilon^2)$ as $\epsilon\to 0$

$=H(1+2\mu\epsilon)+O(\epsilon^2)$ as $\epsilon\to 0$ ……….. (1)

Let $A=\int_\epsilon Hdudv$

where A is the area of the surface enclosed by ' c '.

So,

$$A^{*}=\int_\epsilon H^{*}dudv$$

$$=\int_\epsilon [H(1-2\mu\epsilon)+O(\epsilon^2)]dudv$$

$$=\int_\epsilon Hdudv-\int_\epsilon 2\epsilon\mu Hdudv+O(\epsilon^2) \quad \ldots\ldots\ldots\ldots (2)$$

Since A is stationary in the R.H.S of (2) there should no term containing ϵ

(ie) $2\mu\int_\epsilon Hdudv=0.$

$\Rightarrow\ \mu=0$

Mean curvature is zero.

Problem 1:

Prove that the asymptotic lines on a maximal surface are orthogonal.

Solution:

We know that, If the two directions are given by, $Pdu^2+2Qdudv+Rdv^2=0$

are orthogonal if $ER-2FQ+GP=0$ …………(1)

Let us the Surface be minimal the differential equations giving the asymptotic directions are

$Ldu^2+QMdudv+Ndv^2=0.$

These direction are orthogonal

$\Leftrightarrow EN-2FM+GL=0$

$\Leftrightarrow \frac{EN-2FM+GL}{2(EG-F^2)}=0$

$\Leftrightarrow$ mean curvature $=0$

$\Leftrightarrow$ the surface is minimal.

4.8.Ruled Surfaces:

A Ruled surface is generated by the motion of the straight line moving with one degree of freedom. The varies position of the line are called generating lines or ruling.

Example:

Cones, cylinder, coincides are spherical forms of Ruled surfaces.

To find the equation of the Ruled Surfaces

Let C be a base curve on a given ruled surface then the surface is determined by,

(i) the base curve

(ii) The direction of the generator at the point of the meeting with the curve.

Let $\hat{g}(u)$ be the unit vector along the generate at a curved point Q and C and $r(u)$ be the position vector of Q.

Then $\bar{R}$ be the position vector of the general Pt P is given by $\bar{R} = \bar{r} + v\hat{g}$.

where V is the parameter which measures the directed distance along the generator from C.

To find the metric, unit, normal and the 10^{m} Second fundamental form to a ruled Surface:

Equation of the Ruled Surface is

$\bar{R} = \bar{r} + v\hat{g}$ .... (1)

$\overline{R_1} = \dot{r} + v\dot{\bar{g}}$

Diff w.r.to u is denoted by the suffix 1

Diff (1) with r aspect to v

$\overline{R_2} = \hat{g}$

$$E = \bar{R}_1 \cdot \bar{R}_1$$
$$= (\dot{r}_1 + v\dot{\bar{g}}) \cdot (\dot{\bar{r}} + v\dot{\bar{g}})$$
$$= \dot{\bar{r}}\dot{\bar{r}} + 2v\dot{\bar{g}}\dot{\bar{r}} + \dot{g}\dot{\bar{g}}v^2$$
$$G = \bar{R}_2 \cdot \bar{R}_2 = \hat{g} \cdot \hat{g} = 1$$
$$F = \bar{R}_1 \cdot \bar{R}_2 = (\dot{\bar{r}} + v\dot{\bar{g}})(\dot{\bar{r}} + v\dot{g}). \text{ [Diff } \hat{g}\dot{\bar{g}} = 0 \text{]}$$
$$= \dot{\gamma}\hat{g} + v$$
$$= \dot{\bar{r}}\hat{g} - \int du^2 + 2\dot{\bar{r}}\hat{g}dudv + dv^2$$

Thus the metric is

$ds^2 =$

unit norm: $-(\dot{\vec{r}} + v\dot{g}) \times \hat{g}$ ....(2)

Second Fundamental co-efficients:

$HL = [\overline{R_{11}}, \overline{R_1}, \overline{R_2}]$
$= [\ddot{\gamma} + v\ddot{g}\dot{\gamma} + v\dot{g}\hat{g}]$
$= [\ddot{\gamma}\dot{\gamma}\hat{g}] + v[\dot{g}\dot{\gamma}\hat{g}] + v[\ddot{\gamma}\dot{\gamma}\hat{g}]$
$+v^2[\ddot{g}\dot{g}g]$
$HM = [\bar{R}_{12}, \bar{R}_1, \bar{R}_2]$
$= [\ddot{g}, \dot{\gamma} + v\ddot{g}\hat{g}]$
$= \left[\frac{\ddot{g}}{\dot{\gamma}}\bar{g}\right] + v[\ddot{g}, \dot{g}, \hat{g}]$
$= [\dot{g}\dot{\gamma}\bar{g}] + 0$

$HN = [\bar{R}_{22} \quad \bar{R}_1 \quad \bar{R}_2] = 0$

Since $\bar{R}_{22} = 0$.

$\Rightarrow N = 0$ Since $H \neq 0$.

Note:

(i)The Gaussian curvature for a ruled Surface is given by

$$= \frac{LN - M^2}{EG - F^2}$$

$$= -\frac{-[\dot{g}\dot{\gamma}\hat{g}]^2}{H^2(EG - F^2)}$$

$$= -\frac{-[\dot{g}\dot{\gamma}\hat{g}]^2}{H^4}$$

$\Rightarrow$ Gaussian curvature for a Ruled Surface is ≤ 0. "The Necessary and Sufficient condition for a ruled Surface to be developable is $[\dot{g}\dot{\hat{\gamma}}\hat{g}] = 0$ " parameter of distribution:-

A function $P(\omega)$ defined by,

$p(u) = \frac{[\dot{\bar{r}}\hat{g}\dot{g}]}{\dot{g}^2}$ ………. (1) is called the parameter of distribution of a ruled surface properties of parameter of distribution:

(ii) The parameter of distribution $P(u)$ is independent of the particular base curve chosen.

By Replacing $\bar{\gamma}$ by $\bar{\gamma} + w\hat{g}$ then parameter of distribution

parameter of distribution w.r.to new base curve$\} = \frac{[\dot{\bar{r}}, + w\dot{g}, \cdot \bar{g}' \dot{g}]}{(\bar{g}')^2}$

$$= \frac{[\dot{\vec{r}}\hat{g}\dot{g}]}{g^2} + \frac{\omega[\bar{r}', \hat{g} \cdot \bar{g}']}{\hat{g}^2}$$
$$= \frac{[\dot{\vec{r}}\hat{g}\dot{g}]}{\dot{\bar{g}}^2}$$
$$= p(u)$$

(iii)The parameter of distribution is independent of choice of the parameter u.

Let I be taken as parameter instead of u so the new $p.0.DD(t)$ is given by

$$P(t) = \frac{\left[\frac{d\bar{r}}{dt}, \hat{g}, \frac{d\bar{g}}{dt}\right]}{\left|\frac{d\bar{q}}{dt}\right|^2}$$

$$= \left[\frac{d\bar{r}}{du}\frac{du}{dt}, \hat{g}, \frac{d\bar{g}}{du}\frac{du}{dt}\right] / \left[\frac{d\bar{g}}{du} \cdot \frac{du}{dt}\right]^2$$

$$= \frac{\left(\frac{du}{dt}\right)^2 / \left[\frac{dr}{du}, \hat{g}, \frac{d\bar{g}}{du}\right]}{\left(\frac{du}{dt}\right)^2 \left[\frac{d\bar{q}}{du}\right]^2} = p(u)$$

In particular are length S is taken as parameter than the parameter of distribution of a generator, $g(S)$ through the point $r(s)$ is given by,

$$p = \frac{[\bar{r}', \hat{g} \cdot \bar{g}']}{(\bar{g}')^2} = \frac{\hat{t}\hat{g}\bar{g}'}{(\bar{g}')^2}$$

(iv)P vanishes identically on a developable Surface, we know that,

The Gaussian curvature for a Ruled Surface is given by,

$$k = \frac{[-\dot{\bar{r}}, \bar{g}, \dot{\bar{g}}]^2}{H^4}$$

$$k = \frac{-p^2 \dot{\bar{g}}^4}{H^4}$$

Thus k is always negative except along those generators $P = 0$ since $k = 0$ for a developable surface we see that p vanishes identically on a developable surface.

UNIT V

Differential Geometry of Surfaces: Compact surfaces whose points are umbilics- Hilbert's lemma – Compact surface of constant curvature – Complete surface and their characterization – Hilbert's Theorem – Conjugate points on geodesics.

Chapter 5: Sections 5.1 to 5.7

5.1. Compact Surfaces Whose Points are Umbilics:

Theorem 1:

The only compact surfaces of class ≥ 2 for which every point is an umbilic are spheres.

Proof:-

Let s be a compact surface of class ≥ 2 for which every point is an umbilic

(i.e.,)points at which $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$

Let P be any point on ' s '.

Let V be a co-ordinate neighbourhood of 's' containing ' P '. in which part of S in represented parametrically by $\bar{r} = \bar{r}(u, v)$

Since every point of ' v ' is an umbilic.

⇒ Every curve is lying in v must be a line of curvature,

Hence, from Rodrigue's formula, at all pts of v is, $kd\bar{r} + dN = 0$ …………..(1)

Where, K = normal curvature of ' S ' in the direction $d\bar{r}$.

From (1)

$$\Rightarrow d\bar{N} = -kd\bar{r}$$
$$\Rightarrow \bar{N}_1 = -k\bar{r}_1 \ \& \ \bar{N}_2 = -k\bar{r}_2$$
$$\Rightarrow \bar{N}_{12} = -k_2 r_1 - k\bar{r}_{12} \ and \ \bar{N}_{21} = -k_1\bar{r}_2 - k\bar{r}_{21}$$

But we have that, $\bar{N}_{12} = N_{21} \ \& \ \bar{r}_{12} = \bar{r}_{21}$

$$\therefore -k_2\bar{r}_1 - k\bar{r}_{12} = -k_1\bar{r}_2 - k\bar{r}_{21}$$
$$\Rightarrow -k_2\bar{r}_1 = -k_1\overline{r_2} \quad [\because \bar{r}_{12} = \bar{r}_{21}]$$
$$\Rightarrow k_2\bar{r}_1 - k_1 r_2 = 0$$

Since r_1, r_2 are L.I

$\Rightarrow k_1 = k_2 = 0$

$\therefore k$ is constant.

Integrate (1), (for $k \neq 0$)

(1) $\Rightarrow \int (d\bar{N} + kd\bar{r}) = 0.$

dr) $\int dN + k \int d\bar{r} = 0$

(i.e.) $\int dN = -k \int d\bar{r}$ $(\because k \neq 0)$

(i.e.) $-k^{-1} \int dN = \int d\bar{r}$

(i.e.) $-k^{-1}N + \bar{a} = \bar{\gamma}$

where $\bar{a} = a$ constant vector (or) $\bar{\gamma} = \bar{a} - k^{-1}N$ ....(2)

⇒ ' V ' lies on the surface of a sphere of center ' a ' and radius k^{-1}.

Integrate (1), for $(k = 0)$

(1) $\Rightarrow \int d\bar{N} + kd\bar{r} = 0$

$\Rightarrow \int d\bar{N} = 0$

$\Rightarrow \bar{N} = \bar{b}$ .... (3)

⇒ ' v ' lies on a plane.

∴ from (2) & (3) $\Rightarrow \bar{r} = \bar{a} - k^{-1}\bar{b}$

The neighborhood of any point the surface is spherical (or) plane local pant of the theorem.

Associate with each point ' p ' on the surface a neighbourhood ' v ' having the above property.

The set of all neighbourhood 's vp covers s & from the compactness we deduce that ' s ' is covered by a finite sub-cover formed by $v_i, i = 1,2 \dots N$.

consider two over lapping neighbourhood's v_i, v_j

from the previous local argument,

⇒ k is constant in v_i and also in v_j.

By considering the value of ' k ' at pt's in $v_i \cap v_j$.

⇒ k takes the same value over the whole of the surface.

Moreover, this value can't be zero. Otherwise, the surface would contain a Straight line & would not be compact.

Hence the surface must be a sphere

5.2. Hilbert's Lemma:

In a closed region R of a Surface of Constant tue Gaussian curvature without umbilics, the principal curvatures take their extreme values at the boundary.

Lemma 1:

If at a point 'P_0 ' of any Surface, the principal curvatures k_a, k_b are such that either (i) $k_a > k_b$. ' k_a ' has a maximum at ' P_0 ' & ' K_b ' has a minimum at P_0.

(or) (ii) $k_a < k_b$, ' k_a ' has minimum at ' P_0 ' & ' K_b ' has a maximum at P_0.

Then the Gaussian curvature k cannot be tue at ' P_0 '.

Proof:

Suppose that the lemma is false

(i.e.) Assume that there is a point, ' p_0 ' at which the principal our value, have distinct extreme. values, one maximum and the other minimum with k at ' p_0 ' is strictly true.

Taking the lines of curvature as parametric curves, we know that, the principal curvatures are,

$$k_a = \frac{L}{E} \quad \dots\dots\dots (1)$$

& $K_b = \frac{N}{G}$ ………… (2)

Also we know that , when the lines of curvature are chosen as parametric curves, the Codazzi relations expressed in terms of E, G, L, N & their derivatives are.

$L_2 = \frac{1}{2} E_2 \left[\frac{L}{E} + \frac{N}{G}\right]$ ………… (3)

$N_1 = \frac{1}{2} G_1 \left[\frac{L}{E} + \frac{N}{G}\right]$ ………… (4)

Now,

Diff (1) with respect to ' v ' (partially)

$$\frac{\partial ka}{\partial v} = \frac{\partial}{\partial v}\left(\frac{L}{E}\right)$$

$$= \frac{EL_2 - LE_2}{E^2}$$

$$= \frac{E\left[\frac{1}{2}E_2\left[\frac{L}{E} + \frac{N}{G}\right]\right]}{E^2} - LE_2 [\text{ by equation (3)}]$$

$$= \frac{\frac{E_2 L}{2} + \frac{EE_2 N}{2G} - LE_2}{E^2}$$

$$= \frac{\frac{EE_2 N}{2G} - \frac{1}{2}LE_2}{E^2}$$

$$\frac{\partial K_a}{\partial r} = \frac{\frac{E_2}{2}\left[\frac{EN}{G} - L\right]}{E^2}$$

$$= \frac{E_2}{2E^2}\left[\frac{EN - GL}{G}\right]$$

$$= \frac{E_2}{2E}\left[\frac{EN - GL}{EG}\right] \qquad \left[\because K_a - K_b = \frac{L}{E} - \frac{N}{G}\right.$$

$$\frac{\partial k_a}{\partial v} = \frac{E_2}{2E}[k_b - k_a] \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$Similarly, \frac{\partial K_b}{\partial u} = \frac{G_1}{2G}[K_a - K_b] \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Since, the principal curvatures, k_a and k_b have extreme value at ' P_0 ',

we have,

$$\frac{\partial k_a}{\partial v} = 0 \;\&\; \frac{\partial k_b}{\partial u} = 0 \text{ at ' } P_0'$$

Sub in (5) & (b)

$$(5) \Rightarrow = \frac{E_2}{2E}[k_b - k_a]$$
$$\Rightarrow E_2[K_b - k_a] = 0, \quad (\because k_a \neq k_b).$$
$$\Rightarrow E_2 = 0\ [\because k_b - k_a \neq 0].$$
$$\& \quad (6) \Rightarrow 0 = \frac{G_1}{2Q}[K_a - K_b].$$
$$\Rightarrow G_1[K_a - k_b] = 0 \quad (\because k_a \neq k_b).$$
$$\Rightarrow G_1 = 0$$

Diff equation (5) with respect to ' v '.

$$\frac{\partial^2 k_a}{\partial v^2} = \frac{1}{2E}[K_b - K_a]E_{22} + \frac{d}{dv}\left[\frac{1}{2E}(K_b - K_a)\right]E_2$$
$$= \frac{1}{2E}[K_b - k_a]\cdot E_{22} \quad \ldots\ldots\ldots(7) \qquad [\because E_2 = 0].$$

Similarly, $\frac{\partial^2 K_b}{\partial u^2} = \frac{1}{2G}[K_a - K_b]G_{11}$ …………..(8)

∴ There are now two possibilities, either (i) k_a has a maximum $(k_a > k_b)$.

then $k_a - k_b > 0$

$\Rightarrow \frac{\partial^2 K_a}{\partial v^2} \leq 0$ & $\frac{\partial^2 K_b}{\partial u^2} \geq 0.$ ……….. (9)

(or) (ii) K_a has a minimum $(k_a < K_b)$

then $k_b - K_a > 0 \Rightarrow \frac{\partial^2 k_a}{\partial v^2} \geq 0$ and $\frac{\partial^2 k_b}{\partial u^2} \leq 0$ …………(10)

∴ using (10) in (7) $\left[k_b - k_a > 0\ \& \frac{\partial^2 k_a}{\partial v^2} \geq 0\right]$

we get,

(7) $\Rightarrow E_{22} \geq 0$

& using (9) in (2) $\left[\because k_a - k_b \geq 0\ \& \frac{\partial^2 k_b}{\partial u^2} \geq 0\right]$.

(8) $\Rightarrow G_{11} \geq 0$

The Gaussian curvature K is.

$$K = \frac{-1}{2H}\left\{\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right\}.$$
$$= \frac{-1}{2H}\left\{G_{11}\frac{1}{H} + G_1\frac{\partial}{\partial u}\left(\frac{1}{H}\right) + E_{22}\frac{1}{H} + E_2\frac{\partial}{\partial v}\left(\frac{1}{H}\right)\right\}$$
$$K = \frac{-1}{2H}\left\{G_{11}\frac{1}{H} + E_{22}\frac{1}{H}\right\} \qquad [\because G_1 \& G_2 = 0].$$

$K = \frac{-1}{2H^2}\{G_{11} + G_{22}\}$ at ' P_0'. ………… (11)

We know that $G_{11} \geq 0$ & $E_{22} \geq 0$.

$\Rightarrow k$ = negative (or) zero.

This is contraction to our assumption.

∴ k cannot be tie at ' P_0 '.

5.3.Compact Surfaces of constant Gaussian or Mean Curvature:

Remarks:

1. A compact surface must passes a 'highest point’ and at this point the curvature is necessarily non-negative.

   ⇒ A compact surface cannot have constant negative curvature.

2. A compact surface cannot have constant zero curvature, for otherwise it would contain straight lines which would contradict the compactness

Theorem 1:

The only compact surface with constant Gaussian curvature are spheres.

Proof:

Let ' s ' be a compact surface with constant positive Gaussian curvature ' k '.

Since ' S ' is compact.

∴ There is a point ' p_0 ' at which the maxi value of the principal curvature is attained.

Since the product of the principal curvatures (ii) The Gaussian (curvature) is constant.

⇒ The principal curvatures have respectively a maximum & a minimum value at ' p_0 ' with the maximum not loss than the minimum.

∴ from Hilbert's Lemma,

The two principal curvature must be equal

(i.e.) At point does either principal curvature exceed $\sqrt{k}$

Hence every point of S is an umbilic .

[The only compact surfaces of class $\geqslant 2$ for Which every pt is an umbilic are spheres].

⇒ The only compact Surfaces with constant Geaussian curvature are sphere.

Theorem 2:

The only compact surfaces whose Gaussian curvature is positive and mean curvature constant are spheres

Proof :

Let S be a compact surface of positive Gaussian curvature and constant mean curvature.

Denote K_a & K_b be large and smaller principal curvatures respectively,

Since ' K_a ' is continuous & ' S ' is compact.

There is a P_0 ' at which K_a attains its maxi value.

Since the mean curvature is constant.

⇒ k_b attains its minimum value at P_0.

[If there is a point P different from ' P_0 ' Such that k_b at P is Smaller than K_b at P_0 then K_a at P is greater than k_a at P_0.

Since the mean curvature is a constant

This is contraction to the maximality of K_a at

Now, we have the relation $M_a > M_b$ everywhere.

If $k_a > k_b$ at ' P_0 ' then $K \leq 0$.

(by Hilbert's Lemma).

This is contraction to our assumption that the Gaussian curvature k is tie.

$\therefore k_a = k_b$ at P_0'

$\therefore$ The mean curvalue $(\mu) = \frac{K_a+K_b}{2}$

$= k_a$ (or) k_b

Hence at every point $'S'$ the mean curvature given by $\mu = \frac{k_a+k_b}{2}$.

[If there is a point ' P ' on ' S ' different from P_0 Such that K_a at ' P ' $> k_b$ at P.

then $\mu = k_a$ at ' P_0 ' $\geq k_a$ at ' P ' $> k_b$ at P

$\geq K_b$ at $P_0 = \mu$

This is contraction.]

$\therefore$ The Gaussian curvature $k = k_a \cdot k_b = \mu^2$ a constant.

Hence 'S' is a compact Surface with constant the Gaussian curvature

$\therefore$ by know theorem $\Rightarrow$ ' S ' is a sphere.

5.4.Complete Surfaces:

Metric Space

A set of points ' S ' carries the structure of a metric space when there is a real-valued f_n.

$\rho: S \times S \rightarrow R$, with the properties.

(i) $\rho(A, B) = 0 \Leftrightarrow A = B$

(ii) $P(A, B) = P(B, A)$

(iii) $\rho(A, C) \leq \rho(A, B) + \rho(B, C), \forall A, B, C$ of S.

Note:-

If S is connected then any two points can be joined by arc-wise connected paths.

Remark:-

The surface can be regarded as a metric space.

Proof:

Assume that the surface S is connected.

$\Rightarrow$ any two pts. can be joined by arc-wise connected paths.

If γ is any path joining A to B, then this path can be divided into a finite no. of segments

$\therefore$ Each segment lies entiverly in one co-ordinate neighbourhood & adjacent co-ordinate neighbourhood is

$$\therefore \left.\begin{array}{l}\text{The length of} \\ \text{The segment}\end{array}\right\} = \int \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt$$

where the co-ordinate are $u = u(t)$ & $v = v(t)$

∴ The length of $\gamma =$ the sum of the lengths of its segments.

Now, we define,

distance function $\rho = \rho(A, B) =$ The greatest lower bound of the lengths of all arc-wise connected C 'paths joining A to B.

This $\rho(A; B)$ Satisfies the conditions (i) (ii) & (iii)

⇒ The surface can be regarded as is positive definite a metric space.

Cauchy Sequence:

A sequence of point's $\{x_n\}$ on the surface is said to form a Cauchy sequence if

given a positive real no. ' ϵ ' an integer ' n_0 '.

Such that $\rho(x_p, x_q) < \varepsilon, p, q$ both exceed n_0.

If $\{x_n\}$ converges to a limit ' x ' then the sequence. $\{x_n\}$ is a Cauchy sequence.

Complete metric space:

If the surface is such that "Every Cauchy sequence converges". Then the metric space is said to be complete.

Example:

Give an example to shows that not all surfaces are complete.

Solution:

Let the surface formed by the two-dimensional Cartesian plane of pairs of real no.'s (x, y). when the origin is removed.

The distance function $'\rho$ ' is the Euclidean distance function defined by,

$\rho(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

When $(x_A, y_A), (x_B, y_B)$ are the rectangular, co-ordinate of point's $A\&B$.

The sequence of points $\left\{\left(\frac{1}{n}, 0\right)\right\}$ is seen to be a cauchy seq. Which does not converge in the Surface.

∴ Surface is not complete.

Note that, the two points $(a, 0), (-a, 0), (a > 0)$ Cannot be joined by a geodesic (Straight line) lying entirely on this surface.

5.5.Characterization of Complete Surfaces:

Theorem 1:

To prove that the following properties are equivalent.

a.) Every Cauchy sequence of points of s is convergent.

b) Every geodesic can be prolonged indefinitely in either direction or else it forms a closed curve

c) Every bounded set of points of ' s ' is relatively compact.

Proof:

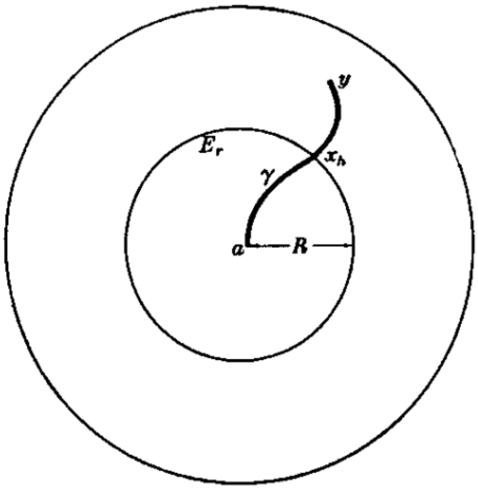

Figure 1.

To prove that $(a) \Rightarrow (b)$

given (a) every Cauchy sequence of points of s is convergent. ………..(1)

To prove that

Every geodesic can be prolonged indefinitely in either direction (or) else it forms a closed curve.

If γ is a closed curve, then every geodesic can be prolonged indefinitely in either direction (or) it forms closed curve.

If γ is not a closed curve and if $p(x)$ is some point on γ .

then there is same no ' l ' such that ν can be prolonged for distance (measured along γ) <1 But, cannot be prolonged for distances $> \ell$

Now, consider the sequence of points $\{x_n\}$ lying on γ at distance from $p($ along $\gamma)$ is $l\left[1-\frac{1}{n}\right]$.

Evidently $\{x_n\}$ is a Cauchy sequence (by (1)) converges to some point Q on γ whose distance from P is precisely ' l '

If $\{x'_n\}$ in another Cauchy sequence, such that $\rho(x_1, x'_n) \to \ell$ then $\{x'_n\} \to Q'$.

Now, the sequence $x_1, x'_1, x_2, x'_2, x_3, x'_3 \cdots$ is also a Cauchy sequence, tending to both $Q \& Q'$

$\therefore Q = Q'$

$\therefore$ their exist a unique and point Q distant l from P along γ .

Now, Consider a coordinate need of S which contains Q

At Q, there is uniquely determined a direction ' t ' which is the direction of the geodesic - γ which starts at Q.

In this coordinate neighbourhood. There is a unique geodesic at Q which has the direction (-t)

$\Rightarrow$ a continuation to our hypothesis (equation (1))

∴ Our assumption is wrong.

⇒ γ is closed curve.

∴ $(a) \Rightarrow (b)$

To prove that $(b) \Rightarrow (c)$.

Given every geodesic can be prolonged indefinitely in either direction or else it forms a closed curve ……….(2)

To prove that : Every bounded set of points of ' S ' in relatively compact.

Consider a point ' a' of ' s '.

and geodesic arcs which start at ' a '

Now we define, Initial vector of a geodesic arc starting at ' a ' to be the tangent vector to tr's arc at ' a '. which has the Same Sense as the geodesic & whose length is equal to the length of the geodesic arc.

Since ' S ' has the property (b),

⇒ Every tangent vector to S at ' a ', its length is the initial vector of some geodesic arc Starting at ' a ' which is uniquely determined.

∴ This arc may eventually cut itself (or) if it forms port of a closed geodesic, may ever cover part itself.

Let S_r be the set of points x of S_r which distance from ' a 'does not exceed r.

(i.e.) $\rho(x, a) \leq r$.

and let E_r be the set of points ' x ' of S_r which can joined to 'a' by a geodesic are whose length is actually equal to $P(x, a)$.

(i) To prove that : The set of points E_r is compact.

Let $\{x_h\}, h = 1,2 \ldots$ be a sequence of points of E_r

& Let T_h be the initial vector of a geodesic an of length $P(a, x_h)$ joining ' a ' to ' x_h '.

Then the sequence of vectors $\{T_h\}$ regarded as a sequence of points in two-dimension Euclidean space admits at least one vector of accumulation T More over,

This vector ' T ' is the initial vector of a geodesic arc whose extremity $\in E_r$& is a $\bar{F}$ accumulation of the sequence $\{x_h\}$.

⇒ E_γ is compact

(ii) To prove that : $E_r = S_r$

$E_r = S_r$ is true when $r = 0$

Also $E_r = S_r$ is true for $r = R > 0$, then it is certainly true for $r < R$.

Now, every pt of S_k is the limit of a sequence of ft^t whose distance from ' a ' $< R$.

By equation, these points $\in E_R$ and since E_R is closed

⇒ Their limit $\in E_\gamma$

∴ $E_r = S_r$ is valid for $r = R$.

$\therefore E_\gamma = S_\gamma$ completely, it is merely to show that it holds for $r = R$, then it still holds for $r = R + S$, $S > 0$

$\Rightarrow$ Because it would then be possible to extend The range of validity of $E_r = S_r$ to an arbitrary extent by an appropriate no. of extensions of the range by an amount ' S '.

To prove that : to any point ' y ' such that $x(a, y) > R$ There is a pt x such that

$\rho(a, x) = R$ & $P(a, y) = R + P(y, x)$

we define,

$P(a, y) =$ The lowest bound of the lengths of arcs from ' a ' to ' y '.

$\Rightarrow$ We can join a to y by a curve γ whose length is less than $\rho(a, y) + h^{-1}$, for any int ' h '. Let $x_h =$ The last point of this curve$\in$ $E_R (= S_R)$

$[We\ know\ that, P(A, c) \leq \rho(A, B) + \rho(B, c)]$

$\therefore P(a, y) \leq \rho(a, x_n) + \rho(x_n, y)$

(i.e.) $P(a, y) \leqslant R + P(x_n, y)$

Since $\rho(a, x_n) = R$

$\Rightarrow$ $P(x_n, y) \geq \rho(a, y) - R - (*)$

Since, the arc length of γ from 'a' to ' y ' = arc length from a to x_n t arc length from x_1 to y . we have,

$P(x_n, y) \leq \mathrm{arc}(x_n, y)$

$$\begin{aligned} \rho(x_n, y) &\leq \mathrm{arc}(a, y) - \mathrm{arc}(a, x_n) \\ \text{(i.e.)} \qquad &\leq \rho(a, y) + h^{-1} - \mathrm{arc}(a, x_n) \\ &\leq P(a, y) + h^{-1} - R \end{aligned}$$

Now let, $h \to \infty$

$\therefore \{x_h\}$ will have at least one point of accumulate x with the property.

$\rho(x, y) \leq \rho(a, y) - R \longrightarrow (**)$

Comparing (*) & (**) we get,

$P(a, y) = R + P(y, x)$

$\therefore$ The existence of a point $'x$ 'satisfying $\rho(a, x) = R$.

$\rho(a, y) = R + \rho(y, x)$

To prove that: Every bounded set of points of s is relatively compact.

we know that, the two points x, y are not for apart, then the point ' y ' in the extremity of one end only one geodesic arc of origin x and length of $\rho(x, y)$.

$\Rightarrow$ their exist a continuous function $s(x) > 0$ such that if $P(x, y) < s(x)$

the point ' y ' is the extremity of the unique geodesic arc of length $\rho(x, y)$ joining x to y. Moreover, the continuous function $s(x)$ attains a tie mini value on the compact set E_R and we take ' s ' to be this minimum

If $E_r = S_\gamma$ is true for $r = R$

and if $R < \rho(a, y) \leq R + s$, then $their\ exists\ x \in E_R$.

Such that $\rho(a, x) = R$, and $\rho(x, y) = \rho(a, y) - R \leq S$.

Consequently,

their exists a geodesic arc L' of length$\rho(a, x)$ joining a to x.

and a geodesic arc L'' of length $\rho(x, y)$ joining x to y.

The composite arc formed by L' & L" joins 'a' to ' y ' and has as its length $\rho(a, y)$.

⇒ This composite arc is a geodesic arc & y is joined to ' a ' by a geodesic arc whose length is equal to the distance of ' y ' from ' a '.

$\therefore y \in E_{R+S}$

and the range of validity of $E_r = S_\gamma$ is extended from E_R to E_{R+S}.

⇒ Any two points of 's' can be joined by a geodesic are whose length is equal to their distance.

Suppose, we use given, a bounded set of pts M on s

5.6.Hilbert's Theorem:

Theorem 1:

A complete analytic surface, free from singularities, with constant negative Gaussian curvature cannot exist in three-dimensional Euclidean Space.

Proof:

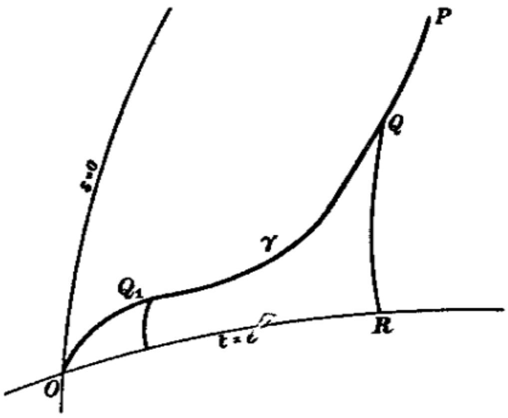

Figure. 2

Let ' p ' be a point on the surface ' S ' & Let ' Q ' be the Set of all paths of ' S ' which begin at ' p '.

we divide the set Q into classes, putting into each class the totality of paths that arc homotopically equivalent.

Let S' is an equivalence class of path on S .Define a natural mapping ϕ of the set S' on the S. (ie) $\phi: S \to S'$).

If ' A ' is a point on S ', then all the equivalent paths in S belonging to A must end in the Same point ' a ' & $a = \phi(A)$.

[Note: The set of points S' can be considered as forming a surface called the "universal covering

Surface"]

"The universal covering surface = S' has the following properties.

1. The natural mapping of S' on S is a continuous open mapping. Moreover, ϕ is a locally homeomorphic mapping. (i.e.) for every point A of S', their exist a neighbourhood e u* Such that the mapping p is homeomorphic on the neighbourhood u^*.

   [The universal covering surface S' of a Surface S is always simply connected.

   (i) ⇒ S & S' are locally homeomorphic.

   ∴ All the local properties of the spaces are automatically true for the Space S']

2. The differential geometric structure on S induces a differential geometric structure on S'

we assume that, a surface 'S' exists having the required properties. Consider an arbitrary geodesic line on the surface S and take an arbitrary point 'o' on as origin. this geodesic If 'S' denotes the arc length of this geodesic measured from 'o'. The completeness of 's' ensures that the geodesic can be continued in both directions from - ∞ to +∞. It is possible that the geodesic will ultimately cross itself have the same point on 'S' will have two different S – values.

5.7. Conjugate points on geodesics:

Theorem 1:

If P and Q are two points of a geodesic which can be embedded in a field of geodesic, then the arc PQ of the geodesic is shorter than any other arc which joins P to Q and lies entirely in that region of the surface covered by the field.

Proof:

The geodesics of the family are the curves v=constant , with $v = v_0$ as the geodesic, and let the curves u=constant be geodesic parallels orthogonal to them, so chosen that the metric reduces to the form

$ds^2 = du^2 + \lambda^2 ds^2$.

If the coordinates of P and Q are $(u_1, v_0), (u_2, v_0)$ with $u_2 > u_1$, the length of the geodesic arc PQ is $(u_2 - u_1)$.

Let C be an arbitrary curve passing through P and Q, is given by the equation $v = \phi(u)$ where $\phi(u_1) = v_0, \phi(u_2) = v_0$. Then the arc length of C is

$$l = \int_{u_1}^{u_2} \left\{1 + \lambda^2 \left(\frac{d\phi}{du}\right)^2\right\}^{1/2} du$$

Evidently l exceeds $u_2 - u_1$, unless $d\phi/du = 0$ when C is the given geodesic.

However, it is most unlikely that the region R of the geodesic field extends over the entire surface S, so the previous argument is in general inapplicable to complete surfaces. For example, the surface of a sphere cannot be covered by a geodesic field because ny two great circles intersect in two points of the sphere. Moreover, if A, B are any two non-antipodal points, that geodesic arc which is he longer part of the great circle joining A, B is evidently not the hortest distance from A to B.

Theorem 2:

When the surface S has negative curvature every. where, the length of a geodesic which joins any two points A, B is aluays less than the lengths of neighbouring curves through A and B.

Proof:

Let one system of parametric curves be the geodesics normal to the given geodesic AB, and the ther system be the orthogonal trajectories. Let u denote the ength of the geodesic normal PQ from P to AB, and let v denote the length AQ. The line element of the surface becomes

$$ds^2 = du^2 + \lambda^2 dv^2,$$

where $\lambda(0, v) = 1, \lambda_1(0, v) = 0.$

In terms of these parameters the Gaussian curvature is given by

$K = -\lambda_{11}/\lambda,$ so that $\lambda_{11} = -\lambda K.$

The function λ may thus be expanded as a power series in u in the form

$$\lambda = 1 - K\frac{u^2}{2} - K_1\frac{u^3}{6} + O(u^4)$$

where K and K_1 are evaluated with $u = 0$.

A neighbouring curve APB which differs very little from AB will have an equation of the form $u = \phi(v)$, where u will be small. The length of this curve will be

$$l = \int_A^I \{\phi'^2 + \lambda^2\}^1 dv = \int_A^B \left\{1 + \phi'^2 - K\phi^2 - \frac{1}{3}K_1\phi^3\right\}^1 dv$$

where terms of the fourth order are neglected. We now assume that ϕ' never becomes infinito and is thus of the same order of smallness as u. With this assumption the difference between l and the geodesic are length s may be written

$$l - s = \frac{1}{2}\int_A^B \left\{\phi'^2 - K\phi^2 - \frac{1}{3}K_1\phi^3\right\} dv$$

Now the sign of the variation of the are length will be given by the $\sec^2$ nd-order terms, provided that these do not vanish identically. If only these terms are retained the equation becomes $l - s = \frac{1}{2}\int_A^B \ (\phi'^2 - K\phi^2) dv$ ……………(1)

Now, if K is always negative, the integrand is always positive and so $l > s$. This proves the required result.

The remainder of this section will consider the analogous problem when K is not always negative. Since the metric is of the form $ds^2 = du^2 + \lambda^2 dv^2$, it follows that the arc length of the orthogonal trajectory taken between the geodesics v and $v + \delta v$ is given by $\lambda \delta v$. Alternatively, $\lambda \delta v$ is the length of the segment of the normal from a typical point of the geodesic v cut off by the geodesic $v + \delta v$. If v and $v + \delta v$ are regarded as constants, then the arc length $\lambda \delta v$ will vary with the arc length u of the geodesic v. If $p = \lambda \delta v$, from

$\lambda_{11} = -K\lambda$ it follows that $p_{11} = -Kp$, (i.e.) $d^2p/du^2 + Kp = 0$

a differential equation which was first obtained by Jacobi in 1836. Consider the solution of this differential equation which vanishes at the point A, and suppose that this solution vanishes again at the point A_1 on the geodesic, while maintaining a constant sign in the interval AA_1. Then all the geodesics which leave A in a direction infinitesimally near to the direction of AB will intersect AB again in the point A_1 or in points infinitesimally near A_1. Now if B lies between A and A_1, it follows that the geodesic segment AB is shorter than any neighbouring curves joining A and B. The point A_1 is called a conjugate point of A along the geodesic A, B.

Theorem 3:

In order that the geodesic distance AB should be the shorlest distance, it is necessary and sufficient that B lics between A and its conjugate point A_1.

The sufficiency has been proved above. We now outline a proof of the necessity using a lemma due to Erdmann.

consider the problem of finding a curve $y = y(x)$, which pasees hrough two pints $(x_1, v_1), (x_2, z_2)$, has a discontinuity of slope on the lome $x = x_3$, and is stact that the integral

$$I = \int_{x_1}^{T_1} f(x, y, y')dx$$

assumes an extreme value (see Fig. 3).

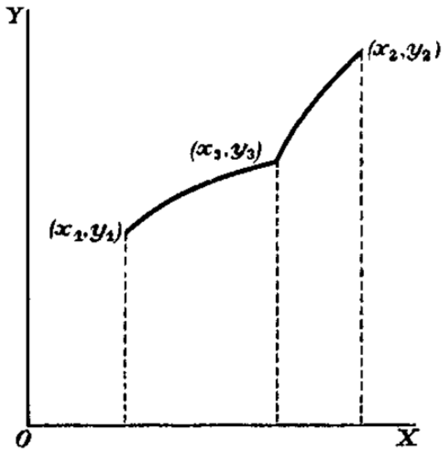

Fig.3

Let $y'_+ = \lim_{\delta \to 0} y'(x_3 + \delta)$; $y'_- = \lim_{\delta \to 0} y'(x_3 - \delta)$

where δ is positive. Then Erdmann's lemma states that for an cxtreme value, in addition to the equation of Euler, it is necessary that

$f_{+y'} = f_{-y'}$

where $f_{+y'} = f_{y'}(x_3, y_3, y'_+), f_{-y'} = f_{y'}(x_3, y_3, y'_-)$.

To prove the lemma, we note that the variation of the integral over the curves $y(x)$ and $y + \epsilon\eta(x)$, where $\eta(x_1) = 0, \eta(x_2) = 0$, is given by

$$J(c) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta')dx + \int_{z_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta')dx$$

it being assumed that the 'corner' still moves along the line $x = x_3$. In the usual manner, it follows that a necessary condition is $J'(0) = 0$. This reduces to

$$\int_{x_1}^{x_2} \left(f_y - \frac{d}{dx} f_{y'}\right) \eta dx + \int_{x_2}^{x_1} \left(f_y - \frac{d}{dx} f_{y'}\right) \eta dx + \eta_3 (f_{-y'} - f_{+y'}) = 0.$$

From this it follows that, in addition to Euler's equation

$f_y - df'_{y'} dx = 0$, it is necessary to have $f_{+y'} = f_{-y'}$, and the lemma is proved.

We now return to the proof of Theorem 3. From equation (1). it follows that the geodesic distance s is a manimum provided that

$$\delta^2(s) = ! \int_{-1}^{1} (u'^2 - Ku^2)du$$

is non-negative. Now, if $\delta^2(s) \geq 0$ for all u, it follows that the curve $u = 0$ must make the integral

$$\int_A^B (u'^2 - Ku^2)dv$$

minimum. It is easily verified that, except for notation, the Euler equation corresponding to this is Jacobi's differential equation.

Assume now that the geodesic distance AB still gives the shortest distance with B lying beyond A_1, i.e. $\delta^2(s) \geqslant 0$, and we hope to arrive at a contradiction. By hypothesis there is a solution of Jacobi's differential equation (and therefore of Euler's equation) which vanishes at A, and has its next zero at A_1. If $u = \phi(v)$ is such a solution, then, of course, so is $u = \epsilon\phi(v)$ for an arbitrary constant ϵ.

Now define a new function $\tilde{u}$ which coincides with $u = \phi(v)$ from A to A_1, and is identically zero from A_1 to B. The next step in the argument is to prove that such a function $\tilde{u}$ is a 'corner' solution of the problem of giving $\delta^2(s)$ an extreme value.

Since $\int_A^{A_1} uu''dv = [uu']_A^{A_1} - \int_A^{A_1} u'^2dv = -\int_A^{A_1} u'^2dv,$

where $u = \phi(v)$, it follows that

$\int_A^{n} (\pi'^2 - K'\bar{u}^2)dv = \int_A^{A_1} (u'^2 - Ku^2)dv = -\int_A^{A!} u(u'' + Ku)dv = 0,$

since $u'' + Ku = 0$.

Since $\tilde{u}$ satisfies the condition $\delta^2(s) = 0$, and can be chosen as near to the curve $u = 0$ as we please since ϵ is arbitrary, it follows that $u = 0$ gives $\delta^2(s)$ its minimal value. Moreover, u must be a 'corner' solution of the problem of finding a minimum of $\delta^2(s)$. From Erdmann's lemma, $u'_+ = u'_- = 0$. But this is impossible because there is no non-trivial solution of the equation $u'' + Ku = 0$

which vanishes simultaneously with its derivative. This gives If_8 required contradiction, and the theorem is completely proved.

Jacobi's theorem will now be used to prove the following interesting theorem due to Bonnet.

Theorem 4:

If along a geodesic the Gaussian curvature exceeds a positive constant $1/a^2$, then the curve cannot be the shortest distance between its extremities along an are length exceeding πa.

The main lemma used in the proof of this result is a standard theorem from the theory of differential equations due to Sturm This theorem is stated below without proof, but a very simple and elegant proof can be found in Darboux (1896).

STURM'S Theorem.

Consider the two distinct differential equations

$$\frac{d^2V}{dx^2}=HV, \frac{d^2V}{dx^2}=H'V$$

where for all values of x in the range considered, $H'(x) \geqslant H(z)$. Then, if $\phi(x)$ is a solution of the first equation hating two consecutive zeros at x_0 and x_1, a solution of the second equation which has a zero at x_0 cannot hate another zero in the closed interval $[x_0, x_1]$.

As a corollary we have:

If for all values of x in the range considered, $H'(x) \leqslant H(x)$, and if $\phi(x)$ is a solution of the first equation having two consecutive zers at x_0 and x_1, then any solution of the second equation which has a zero at x_0 must hare at least one other zero in the inverval $[x_0, x_1]$.

Consider Jacobi's differential equation $(d^2p/dv^2) + Kp = 0$, which is of the type considered by Sturm. Let p be a solution of thas equation, and let r_0, v_1 be two consecutive zeros corresponding to the points A and A_1. It follows from Jacobi's theorem that the arc AB will be the shortest distance between A and B if and only if B lies between A and A_1.

Suppose now that the Gaussian curvature along the line AA_1 always exceeds the positive constant $1/a^2$, so that $K \geqslant 1/a^2$. The solution of the equation

$$\frac{d^2p}{dv^2}=-\frac{p}{a^2}$$

which vanishes for $v = v_0$ is

$$C\sin\frac{v-v_0}{a}$$

and its next zero after v_0 is just $v_0 + \pi a$. It follows that if the are length AB exceeds a, then B will not lie between A and A_1, and the theorem is proved.

An analogous result is the following:

Theorem 5:

If at all points of a geodesic the Gaussian curvature is less than $1/b^2$, the curve is necessarily of shorter length than neighbouring curves along an are length at least equal to πb.

The proof follows easily from the hypothesis $K \leqslant 1/b^2$, and the fact that the interval between consecutive roots of the equation $d^2p/dv^2 = -p/b^2$ is πb. As this cannot be smaller than the interval between consecutive roots of the previous equation, it follows in this case that if they are length AB is less than πb, then B will certainly lie between A and A_1, thus giving the required result.

Suppose now the surface S is compact, and has the property that $K \geqslant 1/a^2$ everywhere. If A and B are any two points on S, there is a geodesic joining A to B which is of shorter length than the neighboring curves.

It follows from Theorem 4. that the maximum distance between A and B cannot exceed πa. This proves the following:

Theorem 6:

If on a compact surface S the curvature everywhere exceeds $1/a^2$, the maximum distance between any two points cannot exceed πa.

Exercise 1:

Prove that the Gaussian curvature at any point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given by $\frac{p^4}{a^2b^2c^2}$ where p is the distance of the centre from the tangent plane. Show that if $a \geqslant b \geqslant c$, every geodesic arc of length greater than $\pi ab/c$ cannot be the shortest distance between its extremities; but every geodesic arc of length less than $\pi bc/a$ is necessarily shorter than the neighbouring curves joining its extremities.

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